# THE STRUCTURE OF COMPLETE EMBEDDED SURFACES WITH CONSTANT MEAN CURVATURE

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## 0. Introduction

We consider complete, properly embedded surfaces $\Sigma \subset \mathbf{R}^3$ which are of finite topological type and have constant nonzero mean curvature. Besides the round sphere the simplest such $\Sigma$ are the noncompact, periodic surfaces of revolution discovered by Delaunay. The main result of this paper is that for each annular end $A \subset \Sigma$ there is a Delaunay surface $D \subset \mathbf{R}^3$ to which $A$ converges exponentially as $|\mathbf{x}| \to \infty$. This means $\Sigma$ itself is conformally a compact Riemann surface having finitely many punctures. As a preliminary step we prove that Delaunay surfaces are the *only* two-ended $\Sigma$. We also derive and use a "balancing formula". It implies, for example, that each end of $\Sigma$ has a "weight vector" parallel to the axis of its limiting Delaunay surface and that the sum of these weights is **0**.

Two recent papers stimulated this research. In [9] W. Meeks proved that any annular end of $\Sigma$ is contained in a solid half-cylinder of some finite radius, that no one-ended $\Sigma$ exist and that two-ended $\Sigma$ are contained in solid cylinders. In [7] N. Kapouleas constructed a wealth of (immersed and embedded) constant mean curvature surfaces $\Sigma$ by solving an elliptic singular perturbation problem. His examples all have asymptotically Delaunay ends. Also, to construct suitable initial surfaces for his perturbation technique he required an approximate "balancing condition" (implied by our balancing formula) to hold for his configuration.

Our paper is organized into six sections. In §1 we review Meeks' results: we significantly simplify his proof of the cylindrical-boundedness theorem, and list some lemmas from his work which we need. §2 contains a systematic treatment of the Alexandrov reflection technique [1], applied to noncompact surfaces. On any cylindrically-bounded end we

show that a certain measure of axial symmetry improves as $|\mathbf{x}| \to \infty$, allowing us to prove that two-ended $\Sigma$ are Delaunay surfaces, and later, in §5, to study Jacobi fields in connection with the strong-convergence theorem. In §3 we use the first variation formula to derive several integral identities and estimates: the balancing formula, a linear area growth estimate for cylindrically-bounded ends, and a version of the monotonicity formula. We also introduce the concept of weights and discuss their properties. In §4 we use these facts and the Gauss-Bonnet formula to argue that the second fundamental form of $\Sigma$ is uniformly bounded. (Parts of this argument are in the spirit of the compactness theorem of H. I. Choi and R. Schoen [2].) §5 contains a proof of the strong-convergence result for annular ends, using the results from §§1–4 and finally a Jacobi field analysis (following ideas of L. Simon [14] and of Simon and B. Solomon [16]). In §6 we offer some concluding remarks and open questions.

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We set the notation and conventions which we will use in different sections of the paper.

**(0.1)** We identify vector fields and their component functions taken with respect to the *standard orthonormal frame* $(\mathbf{e}_1, \cdots, \mathbf{e}_{n+1})$ on $\mathbf{R}^{n+1}$. For example, the components of the *position vector field* $\mathbf{x}$ are the *standard coordinates* $(x^1, \cdots, x^{n+1})$ on $\mathbf{R}^{n+1}$.

A *(hyper)surface* $\Sigma \subset \mathbf{R}^{n+1}$ will be complete, connected, and properly embedded. Consequently, $\Sigma$ separates $\mathbf{R}^{n+1}$ into two connected components, and we can choose a global *unit normal* $\nu$ to $\Sigma$. If $\mathbf{Y}$ is any vectorfield along $\Sigma$, we let $\mathbf{Y}^\perp = (\mathbf{Y} \cdot \nu)\nu$ and $\mathbf{Y}^\top = \mathbf{Y} - \mathbf{Y}^\perp = (\mathbf{Y} \cdot \mathbf{f}_i)\mathbf{f}_i$ (summation convention) denote its *normal* and *tangential projections*, respectively. (Here $(\mathbf{f}_1, \cdots, \mathbf{f}_n)$ is any *local orthonormal frame* on $\Sigma$.)

We let D denote the standard *covariant differentiation on* $\mathbf{R}^{n+1}$. The *divergence on* $\mathbf{R}^{n+1}$ of any (smooth) vectorfield $\mathbf{Y}$ is given by $\mathrm{DIV}\,\mathbf{Y} = (\mathrm{D}_{\mathbf{e}_i}\mathbf{Y}) \cdot \mathbf{e}_i$. Similarly, the *divergence on* $\Sigma$ is defined by $\mathrm{div}\,\mathbf{Y} = (\mathrm{D}_{\mathbf{f}_i}\mathbf{Y}) \cdot \mathbf{f}_i$. The *gradient (covariant derivative)* and *Laplacian on* $\Sigma$ are denoted by $\nabla u = (\mathrm{D}u)^\top$ and $\Delta u = \mathrm{div}(\nabla u)$, respectively.

The *mean curvature vector* of $\Sigma$ is $\mathbf{h} \equiv -h\nu \equiv (\mathrm{D}_{\mathbf{f}_i}\mathbf{f}_i)^\perp = \Delta\mathbf{x}$. Thus (up to sign) the *mean curvature* $h$ is the trace of the *second fundamental form* $\mathbf{A}$ of $\Sigma$. If $h$ has the constant value $H$ we call $\Sigma$ an *MCH-surface*. We can (and will) assume $H \geq 0$ and (if $H > 0$) will often rescale $\Sigma \subset \mathbf{R}^{n+1}$ so that $H = 1$.

In analogy with compact surfaces we call the component $\Omega$ of $\mathbf{R}^{n+1} \backslash \Sigma$ into which $\mathbf{h} = -H\nu$ points the *interior*. Thus our sign convention means that $\nu$ is the *exterior normal* to $\Sigma = \partial\Omega$.

(0.2)   For $0 < R < \infty$ and $\mathbf{P} \in \mathbf{R}^{n+1}$, let $\mathbf{B}_R^{n+1}(\mathbf{P}) = \mathbf{B}_R(\mathbf{P})$ be the *ball of radius $R$ and center $\mathbf{P}$*, $\mathbf{B}_R(\mathbf{P}) = \{\mathbf{y} \in \mathbf{R}^{n+1} : |\mathbf{y} - \mathbf{P}| < R\}$. Its *boundary sphere* is denoted by $\mathbf{S}_R^h(\mathbf{P}) = \mathbf{S}_R(\mathbf{P}) = \partial\mathbf{B}_R(\mathbf{P})$. Given a unit vector $\mathbf{v}$, the (*equatorial*) *disc with center $\mathbf{P}$ and normal $\mathbf{v}$*, is defined by

$$\mathbf{D}_{\mathbf{v},R}(\mathbf{P}) = \{\mathbf{y} \in \mathbf{R}^{n+1} : |\mathbf{y} - \mathbf{P}| \leq R, (\mathbf{y} - \mathbf{P}) \cdot \mathbf{v} = 0\}.$$

The *solid cylinder* generated by $\mathbf{D}_{\mathbf{v},R}(\mathbf{P})$ and $\mathbf{v}$ is $\mathbf{C}_{\mathbf{v},R}(\mathbf{P}) = \{\mathbf{y} + x\mathbf{v} : \mathbf{y} \in \mathbf{D}_{\mathbf{v},R}(\mathbf{P}), x \in \mathbf{R}\}$. The *solid half cylinder* is $\mathbf{C}_{\mathbf{v},R}^+(\mathbf{P}) = \{\mathbf{y} + x\mathbf{v} : \mathbf{y} \in \mathbf{D}_{\mathbf{v},R}(\mathbf{P}), x > 0\}$. If the center $\mathbf{P}$ of a ball, sphere or disc is not specified it is assumed to be $\mathbf{0}$. If its radius $R$ is not specified it is assumed to be 1.

A subset $S \subset \mathbf{C}_{\mathbf{v},R}(\mathbf{P})$ is said to be the *graph of a function $u: \mathbf{D}_{\mathbf{v},R}(\mathbf{P}) \to \mathbf{R}$* if $S = \{\mathbf{y} + u(\mathbf{y})\mathbf{v} : \mathbf{y} \in \mathbf{D}_{\mathbf{v},R}(\mathbf{P})\}$.

We occasionally consider the *sum* of sets $M, N \subset \mathbf{R}^{n+1}$ defined by $M + N = \{\mathbf{m} + \mathbf{n} : \mathbf{m} \in M, \mathbf{n} \in N\}$. For example, $\mathbf{D}_{\mathbf{v},R}(\mathbf{P}) = \mathbf{P} + \mathbf{D}_{\mathbf{v},R}$.

(0.3)   In general we denote an *open subset* of $\Sigma$ by $S$. Often there is an open subset $U \subset \mathbf{R}^{n+1}$ having piecewise smooth boundary and the property that $S = \partial U \cap \Sigma$. In this case we call $Q = \partial U \backslash \Sigma$ a *cap* for $(S, U)$.

A subset $E \subset \Sigma$ is called a *cylindrically-bounded end* if there are a corresponding half cylinder $\mathbf{C}_{\mathbf{v},R}^+(\mathbf{P})$ and an open subset $W = \Omega \cap \mathbf{C}_{\mathbf{v},R}^+(\mathbf{P})$ such that $E = \partial W \backslash \mathbf{D}_{\mathbf{v},R}(\mathbf{P})$.

An *annular end* $A \subset \Sigma \subset \mathbf{R}^{n+1}$ is a properly embedded subset homeomorphic to the *punctured unit disc* $\mathbf{D} \backslash \mathbf{0}$ of $\mathbf{R}^n$. We use $\mathbf{F}$ to denote the homeomorphism from $\mathbf{D} \backslash \mathbf{0}$ to $A$, with $\mathbf{F}(\mathbf{y}) \to \infty$ as $\mathbf{y} \to \mathbf{0}$. More generally we use $\mathbf{F}$ for *parametrizations* of subsets of $\Sigma$.

We say $\Sigma$ is of *finite type* if it is homeomorphic to a closed (i.e., compact, without boundary) manifold with a finite number of closed submanifolds deleted. If $\Sigma$ is two-dimensional and of finite type, then each end of $\Sigma$ is necessarily annular.

## 1.   Meeks' Theorem: cylindrical-boundedness of annular ends

In [9] Meeks proved that a properly embedded MC1-annulus $A = \mathbf{F}(\mathbf{D} \backslash \mathbf{0}) \subset \mathbf{R}^3$ must be *cylindrically-bounded*. The proof of the key lemma there used two difficult analytic results, namely, the construction of an auxiliary embedded minimal disk and a curvature estimate for that disk. In this section we simplify the proof of his Lemma (1.5), using a linking argument to

avoid the minimal surface construction. For completeness, we also outline the proof of Meeks' Theorem.

First, we recall a basic concept from low-dimensional topology. Given two disjoint oriented simple loops $\gamma$ and $\delta$ in $\mathbf{R}^3$, let $\Delta \subset \mathbf{R}^3$ be an oriented surface transverse to $\gamma$ with $\delta = \partial\Delta$. Then the *linking number* $\mathbf{lk}(\gamma, \delta)$ and *intersection number* $\mathbf{i}(\gamma, \Delta)$ are related as follows (cf. [11]):

$$(1.1) \qquad\qquad \mathbf{lk}(\gamma, \delta) = \mathbf{i}(\gamma, \Delta).$$

Here $\mathbf{i}(\gamma, \Delta)$ is the number of points in $\gamma \cap \Delta$, counted with a sign depending on the relative orientation at each intersection point. We take (1.1) as the definition of linking number. Observe that $\mathbf{lk}$ is *symmetric* ($\mathbf{lk}(\gamma, \delta) = \mathbf{lk}(\delta, \gamma)$); also

$$(1.2) \qquad\quad \mathbf{lk}(\gamma, \delta) = 0 \quad \text{if } \gamma \text{ and } \delta \text{ are unlinked,}$$

that is, *if there exists $\Delta$ as above with $\gamma \cap \Delta = \varnothing$*. An easy generalization of (1.2) asserts that *if $\gamma$ and $\tilde\gamma$ are homologous ($\gamma \approx \tilde\gamma$) in $\mathbf{R}^3 \backslash \delta$, then*

$$(1.3) \qquad\qquad \mathbf{lk}(\gamma, \delta) = \mathbf{lk}(\tilde\gamma, \delta).$$

This lets us view $\mathbf{lk}(\cdot, \delta)$ as a *cohomology class in $H^1(\mathbf{R}^3 \backslash \delta, \mathbf{Z}) \cong \mathbf{Z}$* (it is actually a *generator* [11]): *if we represent a homology class by $\gamma \approx \sum_{i=1}^{k} g^i \gamma_i$ for simple loops $\gamma_i \subset \mathbf{R}^3 \backslash \delta$, then*

$$(1.4) \qquad\qquad \mathbf{lk}(\gamma, \delta) = \sum_{i=1}^{k} g^i \mathbf{lk}(\gamma_i, \delta).$$

We can now give our proof of Meeks' key lemma.

**(1.5) Lemma** (Plane-separation). *Let $A \subset \mathbf{R}^3$ be a properly embedded annulus with (possibly varying) mean curvature $h \geq 1$. Let $\pi_+$ and $\pi_-$ be (parallel) planes in $\mathbf{R}^3$ with $\mathrm{dist}(\pi_+, \pi_-) > 4$. Let $\Pi_+$ be the closed half-space bounded by $\pi_+$ and disjoint from $\pi_-$; define $\Pi_-$ similarly. Then at least one of $A \cap \Pi_+$ or $A \cap \Pi_-$ has only compact connected components.*

*Proof.* Suppose, on the contrary, both $A \cap \Pi_+$ and $A \cap \Pi_-$ contain *noncompact* components. Meeks' idea [9] was to use this hypothesis to find a large round sphere (with mean curvature $h < 1$) lying "inside" $A$, contradicting the mean curvature comparison principle [3] when the sphere is moved to first contact with $A$.

We may assume (by applying a rigid motion) that $x^3 \equiv \pm z$ on $\pi_\pm$ for some $z > 2$. For $R > z$ such that $\partial A \subset \mathbf{B}_R$, let $\mathscr{O}$ denote the (closed) solid torus which forms a tubular neighborhood of radius $(z+2)/2$ about the core circle $c = \{x^3 = 0\} \cap \mathbf{S}_{R+z}$, and let $\mathscr{T} = \partial\mathscr{O}$ be its boundary torus. (We will use the comparison argument above with a sphere moving

through $\mathcal{O}$ toward the "inside" of a particular component of $A \cap \mathcal{O}$.) By Sard's theorem and choice of $R$ we may assume that $A \cap \mathcal{T}$ consists of disjoint simple loops.

(i) Let $\lambda$ and $\mu$ denote respectively the *latitude* and *meridian* curves in $\mathcal{T}$; viewed as curves in $\mathcal{O}$, this means $\lambda \approx c$ and $\mu$ is trivial. Recall (or see [11]) that *any (essential) simple loop in $\mathcal{T}$ is homologous (in $\mathcal{T}$) to $l\lambda + m\mu$ for some (relatively prime) integers $l$ and $m$; in particular, if $l = 0$ then $m = \pm 1$.*

(ii) *For $R$ sufficiently large there is a simple loop $\delta \subset A \backslash \mathcal{O}$ which bounds a disc $\Delta \subset A$, and with $\mathbf{lk}(\mu, \delta) = 0$, $\mathbf{lk}(\lambda, \delta) = 1$.*

*Proof of* (ii). The loop $\delta$ is made from four arcs. Construct arcs $\delta_+$, $\delta_-$ in the noncompact components of $A \cap \Pi_+$, $A \cap \Pi_-$, respectively, each beginning on $\mathbf{S}_R$ and approaching $\infty$ (this will determine $R$). Denote by $d_+$, $d_-$ their preimages in $\mathbf{D} \backslash \mathbf{0}$ under $\mathbf{F}$. (Hence $d_+$, $d_-$ approach $\mathbf{0}$.) Since $\partial \mathbf{D} \subset \mathbf{F}^{-1}(\mathbf{B}_R)$ and $\mathbf{F}$ is proper, we may choose $d_0 \subset \mathbf{F}^{-1}(\mathbf{B}_R)$ and $d_\infty \subset \mathbf{D} \backslash \mathbf{F}^{-1}(\mathbf{B}_{3R})$ so that $d = d_+ \cup d_0 \cup d_- \cup d_\infty$ is a simple loop which does not enclose $\mathbf{0}$. Thus $\delta = \mathbf{F}(d)$ is a simple loop, which bounds a disc $\Delta \subset A$, and which is disjoint from the solid torus $\mathcal{O}$. It follows (1.2) that $\mathbf{lk}(\mu, \delta) = 0$.

Let $\mathbf{D}_{R+z}$ be the flat disk with boundary $c$. Then $\mathbf{lk}(\lambda, \delta) = \mathbf{lk}(c, \delta) = \mathbf{i}(\mathbf{D}_{R+z}, \delta_0) = 1$, since $\delta_0 = \mathbf{F}(d_0)$ enters $\mathbf{B}_R$ above $\mathbf{D}_{R+z}$ and exits below.

(iii) *There is a planar domain $\Delta^0 \subset \Delta \cap \mathcal{O}$ with exactly one essential (on $\mathcal{T}$) boundary component $\gamma$, and $\gamma \approx \pm \mu$.*

*Proof of* (iii). Decompose $\Delta \cap \mathcal{T}$ into its components,

$$\Delta \cap \mathcal{T} \approx \sum_{i=1}^{k} \gamma_i.$$

We first show that for each $i$, either $\gamma_i \approx 0$ or else $\gamma_i \approx \pm \mu$ (on $\mathcal{T}$). Writing $\gamma_i \approx l\lambda + m\mu$, it suffices (i) to show $l = 0$. By linearity (1.4) and by (ii)

$$\mathbf{lk}(\gamma_i, \delta) = l\mathbf{lk}(\lambda, \delta) + m\mathbf{lk}(\mu, \delta) = l.$$

But $\gamma_i$ is contractible in the disc $\Delta$ spanning $\delta$, so (1.2) implies $\mathbf{lk}(\gamma_i, \delta) = 0$.

Since $\mathbf{lk}(c, \delta) = 1$ (ii), and because $\Delta \backslash \mathcal{O}$ provides a homology between $\delta$ and $\Delta \cap \mathcal{T} \approx \sum \gamma_i$, at least one of the $\gamma_i \approx \pm \mu$.

Now $\Delta \cap \mathcal{O}$ is (the image under $\mathbf{F}$ of) a finite union of planar domains, with $\partial(\Delta \cap \mathcal{O}) = \bigcup \gamma_i$. From one of these domains pick the innermost $\gamma_i$ satisfying $\gamma_i \approx \pm \mu$. Letting $\Delta^0$ be the component of $\Delta \cap \mathcal{O}$ having this $\gamma_i$ as "outer" boundary, we establish claim (iii).

We now work in the Riemannian universal cover $\tilde{\mathcal{O}}$ of $\mathcal{O}$, which is topologically a solid cylinder, and metrically (locally) Euclidean. Since the

fundamental group $\pi_1(\Delta^0)$ has trivial image in $\pi_1(\mathcal{O})$ (by (iii)), we can choose a compact *lift* $\tilde{\Delta}^0 \subset \tilde{\mathcal{O}}$ of $\Delta^0$. $\tilde{\Delta}^0$ *separates the two ends of* $\tilde{\mathcal{O}}$, for if $\alpha \subset \tilde{\mathcal{O}}$ is any arc running from one end to the other (i.e., $\alpha$ is properly homotopic to the lift $\tilde{c} \subset \tilde{\mathcal{O}}$ of $c$), then $\alpha$ meets $\tilde{\Delta}^0$, because

$$\mathbf{i}(\alpha, \tilde{\Delta}^0) = \mathbf{i}(\tilde{c}, \tilde{\Delta}^0) = \mathbf{lk}(c, \partial \Delta^0) = \mathbf{lk}(c, \pm\mu) = \pm 1.$$

We finish the argument as in [9]. Let a round sphere $\mathbf{S}$ (of radius $r$, with $2 < r < (z+2)/2$ be placed in the component of $\tilde{\mathcal{O}} \backslash \tilde{\Delta}^0$ toward which the mean curvature vector of $\tilde{\Delta}^0$ points. Center $\mathbf{S}$ on $\tilde{c}$, and let $\mathbf{S}$ approach $\tilde{\Delta}^0$ along $\tilde{c}$: $\mathbf{S}$ will first contact at an *interior* point of $\tilde{\Delta}^0$. Since $\tilde{\Delta}^0 \subset \tilde{\mathcal{O}}$ is (locally) congruent to $\Delta^0 \subset A \subset \mathbf{R}^3$, we see that $h \geq 1$ on $\tilde{\Delta}^0$. But $h < 1$ on $\mathbf{S}$, so this first interior contact contradicts the mean curvature comparison principle.

**(1.6) Remark.** Except for the proof of the plane-separation Lemma (1.5) the arguments of this section are $n$-dimensional. Unfortunately, there is a simple annular counterexample to (1.5) (suggested by Meeks) in higher ($n \geq 3$) dimensions: Let $Z \subset \mathbf{R}^3$ be the properly embedded 2-plane obtained by capping a half-cylinder of radius 1 with a matching round sphere; clearly $h \geq 1$ on $Z$. Consider the Riemannian product $\Sigma = Z \times \mathbf{R}^{n-2} \subset \mathbf{R}^{n+1}$, which is a properly embedded $n$-plane with $h \geq 1$. Deleting a small ball from $\Sigma$ we obtain a properly embedded $n$-annulus $A$ which satisfies the hypothesis but not the conclusion of the main lemma; indeed, there are parallel $n$-planes of arbitrary separation which both meet $A$ in noncompact components: any $\mathbf{R}^n$ containing the original $\mathbf{R}^3$ and any parallel copy of this $\mathbf{R}^n$ will meet $\Sigma$ in a properly embedded $(n-1)$-plane.

Of course it would be more interesting if there were an $n$-dimensional annular counterexample to—or proof of—(1.5) for $h \equiv 1$.

The following lemma is used by Meeks. Since we will also need it (in §4) we sketch its proof.

**(1.7) Lemma** (Height-4). *Let* $S \subset \mathbf{R}^{n+1}$ *be a compact MCH-surface with* $\partial S \subset \{x^{n+1} \equiv 0\}$. *Then* $S \subset \{|x^{n+1}| \leq 2n/H\}$.

*Proof.* The usual (compact) Alexandrov reflection argument (cf. §2) reduces the problem to showing that *a bounded MCH-graph* $S = \{x^{n+1} = u \geq 0\}$ *with* $u = 0$ *on* $\partial S$ *satisfies the uniform bound* $u \leq n/H$.

By the Cauchy-Schwarz inequality, the second fundamental form $\mathbf{A}$ and mean curvature $H$ satisfy $n|\mathbf{A}|^2 - H^2 \geq 0$. On a graph, the (upward) unit normal $\nu$ satisfies $\nu^{n+1} \geq 0$. Combining these inequalities with the equations (0.1) $\Delta u = -H\nu^{n+1}$ and $\Delta \nu^{n+1} = -|\mathbf{A}|^2 \nu^{n+1}$ yields the differential inequality $\Delta(Hu - n\nu^{n+1}) \geq 0$ on $S$. Since $Hu - n\nu^{n+1} \leq 0$ on $\partial S$, the

maximum principle implies the same inequality on $S$. The result follows since $\nu^{n+1} \le |\nu| = 1$.

**(1.8) Corollary** [9, 2.3]. *There is no proper, positive coordinate (height) function on a complete, noncompact, properly embedded MCH-surface $\Sigma \subset \mathbf{R}^{n+1}$.*

We proceed to outline the proof of Meeks' Theorem.

**(1.9) Definition** (Axis vector). A unit vector $\mathbf{a}$ is an *axis vector* for $\Sigma \subset \mathbf{R}^{n+1}$ provided there is a sequence of points $\mathbf{p}_i \in \Sigma$ with $|\mathbf{p}_i| \to \infty$ such that $\mathbf{p}_i/|\mathbf{p}_i| \to \mathbf{a}$.

**(1.10) Theorem** (Cylindrical-boundedness [9, 3.1]). *Let $A \subset \mathbf{R}^3$ be a properly embedded $MC1$-annulus. Then there is an axis vector $\mathbf{a}$, a radius $R < \infty$, and a point $\mathbf{p} \in \mathbf{R}^3$ such that $A \subset C_{\mathbf{a},R}^+(\mathbf{p})$.*

*Proof.* Let $\mathbf{a}$ be an axis vector for $A$ as in (1.9). Let $\partial A \subset \mathbf{B}_{R^*}$ and let $\pi = \mathbf{v}^\perp \subset \mathbf{R}^3$ be a plane parallel to $\mathbf{a}$ with $\pi \cap \mathbf{B}_{R^*} = \varnothing$. Pick a family of (slightly) inclined planes $\pi^\varepsilon = (\mathbf{v} - \varepsilon\mathbf{a})^\perp$ with $\pi^\varepsilon \cap \mathbf{B}_{R^*} = \varnothing$ and $\pi^\varepsilon \to \pi$ as $\varepsilon \to 0$. The region "below" $\pi^\varepsilon$ (i.e. containing $\mathbf{B}_{R^*}$) must contain noncompact components because of the height-4 estimate (1.7): a sequence of points $\mathbf{p}_i \in A$ with $|\mathbf{p}_i| \to \infty$ such that $\mathbf{p}_i/|\mathbf{p}_i| \to \mathbf{a}$ satisfies $\mathrm{dist}(\mathbf{p}_i, \pi^\varepsilon) \to \infty$. Hence, by the plane-separation Lemma (1.5) and another application of (1.7), no point of $A$ is more than 8 units above $\pi^\varepsilon$. Letting $\varepsilon \to 0$ we have the same estimate for $\pi$.

Considering all possible planes $\pi$, we find that $A \subset \mathbf{C}_{\mathbf{a},R}$ with $R = R^* + 8$. Since $F$ is proper, $A$ cannot extend to $\infty$ in both directions of this cylinder. In fact, applying (1.7) to the plane (parallel to) $\mathbf{a}^\perp$ and containing the point $-R^*\mathbf{a}$, we conclude that $A \subset \mathbf{C}_{\mathbf{a},R}^+(-(R-4)\mathbf{a})$.

**(1.11) Remark.** The preceding result can be sharpened because of the explicit nature of the estimates (1.5) and (1.7). The separation of $\pi_+$ and $\pi_-$ in Lemma (1.5) can be reduced to 2 as follows: in place of the moving sphere $\mathbf{S}$ of radius $r > 2$, use a torus $\mathbf{T}$ of revolution generated by a circle of radius $r > 1$ about a distant $(> r/(r-1))$ axis (as $r \to 1$, $\mathbf{T}$ approximates a cylinder with $h \equiv 1$). Furthermore, the Alexandrov argument and (1.7) imply that the compact components of $A$ lying outside this slab are graphs of height at most 2. Thus one can prove that *there exists a ball $\mathbf{B} \subset \mathbf{R}^3$ such that $A\backslash\mathbf{B}$ lies in a solid cylinder of radius $R = 3$.* (It is surprising how close this reasoning brings us to the optimal radius bound ($R = 2$) for an asymptotically-Delaunay surface.)

**(1.12) Corollary** [9, Theorems 1, 2, 3]. *Each end of a properly embedded $MC1$-surface of finite type $\Sigma \subset \mathbf{R}^3$ is cylindrically bounded. If $E_1, \cdots, E_k$ are all the cylindrically-bounded ends of $\Sigma$, then the corresponding axis vectors $\mathbf{a}_1, \cdots, \mathbf{a}_k$ cannot all lie in an open hemisphere of $\mathbf{S}^2$. In particular,*

$k = 1$ *is impossible;*

$k = 2$ *implies* $\Sigma$ *is contained in a solid cylinder; and*

$k = 3$ *implies* $\Sigma$ *is contained in a slab.*

*Proof.* The first assertion is immediate, since each end of a 2-dimensional surface $\Sigma$ of finite type is annular (0.3). To prove the second assertion, consider an open hemisphere with pole $\mathbf{n}$; if $\mathbf{a}_1, \cdots, \mathbf{a}_k$ all lie in this hemisphere, then (after a possible translation) the height function $x(\mathbf{p}) = \mathbf{p} \cdot \mathbf{n}$ is positive on $\Sigma$, contradicting (1.8). The final assertions follow from the second by linear algebra.

## 2.  Alexandrov reflection

We study MC$H$-surfaces $\Sigma \subset \mathbf{R}^{n+1}$ that are connected, complete and properly embedded (0.1). Given a plane $\pi$ and its normal vector $\mathbf{v}$, we define a natural *Alexandrov* function with the following property: whenever the function attains an "interior" maximum, $\Sigma$ has $\pi$-parallel plane of reflection symmetry. Focusing on any cylindrically-bounded end $E \subset \Sigma$ (0.3), we prove the key result of this section (2.9): any plane parallel to the cylinder's axis is either parallel to a plane of symmetry for $\Sigma$, or else the corresponding Alexandrov function on $E$ is strictly decreasing in the direction of $\infty$.

One corollary (2.10) is that if $\Sigma$ is contained in a solid cylinder (thus being the union of two cylindrically-bounded ends and *forcing* the Alexandrov function to have an interior maximum), then there must be a plane of symmetry. By considering all planes parallel to the cylinder it follows that $\Sigma$ is rotationally symmetric about a line parallel to the cylinder. In particular, any two-ended $\Sigma \subset \mathbf{R}^3$ must be a Delaunay surface (2.11).

We introduce some notation for the reflection arguments. Fix a hyperplane $\pi \subset \mathbf{R}^{n+1}$ with unit normal $\mathbf{v}$. Let $L$ be the perpendicular line given by $L = \{t\mathbf{v}: t \in \mathbf{R}\}$. For $t \in \mathbf{R}$ and $\mathbf{p} \in \pi$ define the $\pi$-parallel plane $\pi_t$, the closed (upper) half-space $\Pi_t$, and the $\pi$-perpendicular line $L_{\mathbf{p}}$ by

$$(2.1) \qquad \pi_t = \pi + t\mathbf{v}, \quad \Pi_t = \bigcup_{s \geq t}(\pi_s), \quad L_{\mathbf{p}} = \mathbf{p} + L.$$

For any set $G \subset \mathbf{R}^{n+1}$ let $G_t$ be the portion of $G$ above $\pi_t$ and let $\tilde{G}_t$ be its reflection through $\pi_t$:

$$(2.2) \quad G_t = G \cap \Pi_t, \qquad \tilde{G}_t = \{\mathbf{p} + (t - r)\mathbf{v}: \mathbf{p} \in \pi, \ \mathbf{p} + (t + r)\mathbf{v} \in G_t\}.$$

For an open set $W \subset \Omega$ we seek to apply Alexandrov reflection on the surface $S = \partial W \cap \Sigma$ (0.3). Because $S$ is only a subset of $\Sigma$ and because

it may not be connected or bounded we need to be precise about the concept of a (local) first point of reflection contact. For this, and for further discussion in §5, it helps to use the auxiliary function described below.

For a point $\mathbf{p} \in \pi$ consider the line $L_{\mathbf{p}}$ (2.1). If $L_{\mathbf{p}}$ is disjoint from $\bar{W}$ for all sufficiently large $t$, then let $\mathbf{P}_1 = \mathbf{p} + t_1 \mathbf{v}$ be the first point in $L_{\mathbf{p}} \cap \bar{W}$ as $t$ decreases from $\infty$. (When we speak of "first" in reflection arguments we think of $t$ decreasing; the reflection planes are being lowered from above.) If $\mathbf{P}_1 \in S$, if the intersection is transverse and if $L_{\mathbf{p}}$ first leaves $W$ through $S$, call the point where it leaves $\mathbf{P}_2 = \mathbf{p} + t_2 \mathbf{v}$. If $\mathbf{P}_1 \in S$ but the intersection is tangential, let $\mathbf{P}_2 = \mathbf{P}_1$. If $\mathbf{P}_1$ and $\mathbf{P}_2$ both exist (i.e., $L_{\mathbf{p}}$ first enters and leaves $\bar{W}$ through $S$, in the sense above), then $\mathbf{p}$ is in the domain of the *Alexandrov function* $\alpha_1$ defined by

$$(2.3) \qquad\qquad \alpha_1(\mathbf{p}) = \frac{t_1 + t_2}{2}.$$

Note that $\alpha_1(\mathbf{p})$ is the value of $t$ for which the reflection of $\mathbf{P}_1$ through $\pi_t$ first "contacts" $S$, although for nontransverse intersection this "contact" must be suitably interpreted.

**(2.4)** *A local first point of reflection contact* for $S$ with respect to the plane $\pi$ (and normal $\mathbf{v}$) is defined to be a point $\mathbf{P}_2(\mathbf{p})$ for which $\mathbf{p} \in \pi$ is a *local maximum* of $\alpha_1$: There is a neighborhood of points $\mathbf{q}$ near $\mathbf{p}$ so that whenever $\mathbf{q}$ is in the domain of $\alpha_1$, $\alpha_1(\mathbf{q}) \leq \alpha_1(\mathbf{p})$.

**(2.5)** An *interior* local maximum for $\alpha_1$ occurs at $\mathbf{p} \in \pi$ if there is a neighborhood of points $\mathbf{q} \in \pi$ about $\mathbf{p}$ with the property that either $\mathbf{q}$ is in the domain of $\alpha_1$ and $\alpha_1(\mathbf{q}) \leq \alpha_1(\mathbf{p})$, or else $L_{\mathbf{q}} \cap \overline{W} = \varnothing$. Any other local maximum of $\alpha_1$ will be called a local *boundary* maximum (as might occur if $L_{\mathbf{p}}$ intersects $\partial S$).

Definitions (2.4) and (2.5) are justified by the following lemma. It shows that the Alexandrov reflection technique can be applied to noncompact hypersurfaces.

**(2.6) Lemma.** *Fix the plane $\pi$ and its normal $\mathbf{v}$. If, relative to the subsets $S \subset \Sigma$ and $W \subset \Omega$, $\alpha_1$ has a local interior maximum value $z$ at $\mathbf{p} \in \pi$, then the plane $\pi_z$ is a plane of symmetry for $\Sigma$.*

*Proof.* Compare the surface $S$ to the reflection $\tilde{S}_z$ of $S \cap \Pi_z$ through $\pi_z$ (2.2). By construction $\mathbf{P}_1(\mathbf{p})$ reflects to $\mathbf{P}_2(\mathbf{p})$. Let $\mathbf{q} \in \pi$ be near $\mathbf{p}$, in the domain of $\alpha_1$, and with $\mathbf{P}_1(\mathbf{q}) \in \Pi_z$. The local maximality of $\alpha_1(\mathbf{p})$ implies after rearrangement that

$$z - (t_1(\mathbf{q}) - z) \geq t_2(\mathbf{q}),$$

i.e., that the reflection of $\mathbf{P}_1(\mathbf{q})$ through $\pi_z$ lies above $\mathbf{P}_2(\mathbf{q})$, and since $\mathbf{q}$ is in the domain of $\alpha_1$:

(i)                              $\tilde{\mathbf{P}}_1(\mathbf{q})_z \in \bar{W}$.

By (i) a neighborhood of $\tilde{S}_z$ containing $\mathbf{P}_2(\mathbf{p})$ is contained in $\bar{W}$. In particular if $\mathbf{P}_1(\mathbf{p}) \neq \mathbf{P}_2(\mathbf{p})$, $\tilde{S}_z$ and $S$ must be tangent at $\mathbf{P}_2(\mathbf{p})$, with non-vertical tangent plane. In the case $\mathbf{P}_1(\mathbf{p}) = \mathbf{P}_2(\mathbf{p})$, $\tilde{S}_z$ and $S$ are tangent by construction, with vertical tangent plane.

Express $S$ and $\tilde{S}_z$ locally above their common tangent plane (with origin at $\mathbf{P}_2(\mathbf{p})$) as graphs of functions $u$ and $\tilde{u}$. Both functions satisfy the same uniformly elliptic equation (the nonparametric constant mean curvature equation). Both are zero and have zero gradient at the origin. $\tilde{S}_z \subset \bar{W}$ locally (i) implies that near the origin we may take $\tilde{u} \geq u$. Thus by the strong maximum principle [3] $u = \tilde{u}$ locally and $\tilde{S}_z$ coincides with $S$ locally. (Note that one applies the boundary point version in the "vertical" case because (locally) the domain of $\tilde{u}$ is only a half-plane through the origin.) Since $\Sigma$ is real analytic and connected, $\pi_z$ is a global plane of symmetry.  q.e.d.

Lemma (2.6) is useful only if one can find local maxima for $\alpha_1$. This Alexandrov function is *not* continuous, but it is upper semicontinuous and will attain its supremum on any compact domain. In fact there is upper semicontinuity with respect to planes as well as points: Let $S$ be closed. As the parameter $\varepsilon \to 0$, suppose we have a sequence $\mathbf{p}^\varepsilon \to \mathbf{p}$ of points $\mathbf{p}^\varepsilon$, contained in planes $\pi^\varepsilon \to \pi$, and a corresponding sequence of Alexandrov functions $\alpha_1^\varepsilon$. If $\alpha_1^\varepsilon(\mathbf{p}^\varepsilon)$ and $\alpha_1(\mathbf{p})$ exist, then

(2.7)                    $\limsup_{\varepsilon \to 0} \alpha_1^\varepsilon(\mathbf{p}^\varepsilon) \leq \alpha_1(\mathbf{p})$,

for we may assume $\alpha_1^\varepsilon(\mathbf{p}^\varepsilon)$ approaches its lim sup. A subsequence of the corresponding pairs $(\mathbf{P}_1^\varepsilon, \mathbf{P}_2^\varepsilon)$ converges to a pair of (possibly identical) points $(\mathbf{Q}_1, \mathbf{Q}_2)$, each above $\mathbf{p}$ and in $S$. The heights above $\pi^\varepsilon$, $\{t_1^\varepsilon, t_2^\varepsilon\}$, converge to the heights of $\mathbf{Q}_1, \mathbf{Q}_2$ above $\pi$. By definition the points $\mathbf{P}_1, \mathbf{P}_2$ must be at least as high as these, proving (2.7).

Now we consider a cylindrically bounded end $E \subset \mathbf{C}_{\mathbf{a},R}^+(\mathbf{P})$ so that $W = \Omega \cap \mathbf{C}_{\mathbf{a},R}^+(\mathbf{P})$ as in (0.3). Lemma (2.9) below is equivalent to the fact that, given a plane $\pi$ parallel to $\mathbf{a}$, the first point of reflection contact for $E$ must occur on the boundary, i.e. at a reflected point of $E \cap \mathbf{D}_{\mathbf{a},R}(\mathbf{P})$. This claim is surprising because $E$ is unbounded and could a priori have first point of reflection contact "at infinity". We prove the claim by using slightly tilted planes $\pi^\varepsilon$, for which the result is true because behavior at infinity is controlled by cylindrical boundedness. Decreasing the tilt to zero, we obtain the lemma.

Let $\pi$ be a plane parallel to the axis $\mathbf{a}$, with normal $\mathbf{v}$. With a rotation and translation we normalize so that $\mathbf{P} = \mathbf{0}$, $\mathbf{a} = \mathbf{e}_1$, $\mathbf{v} = \mathbf{e}_{n+1}$, and we use $x$

for the first coordinate function, $x(\mathbf{p}) = \mathbf{p} \cdot \mathbf{e}_1$. Let $\alpha_1$ be the corresponding auxiliary function for $E$. We define the related Alexandrov function $\alpha$ on $E$:

$$(2.8) \qquad\qquad \alpha(x) = \max_{\substack{\mathbf{p} \in \pi \\ \mathbf{p} \cdot \mathbf{a} = x \geq 0}} \alpha_1(\mathbf{p}).$$

**(2.9) Lemma.** *For the end $E$ of the complete, properly embedded MCH-surface $\Sigma$ and for any plane $\pi$ parallel to $\mathbf{a}$, either the function $\alpha(x)$ is strictly decreasing, or else $\Sigma$ has a plane of reflection symmetry parallel to $\pi$.*

*Proof.* To prove (2.9) it suffices to show that $\alpha(x)$ is nonincreasing, since then it is either strictly decreasing or constant on some interval. In the second case (2.6) implies the existence of a $\pi$-parallel symmetry plane.

To show $\alpha$ is nonincreasing it suffices to show that $\alpha(x) \leq \alpha(0)$ for all $x > 0$; by translating and redefining the end, our cross-section $x \equiv 0$ may be chosen arbitrarily.

Showing that $\alpha(x) \leq \alpha(0)$ for all $x > 0$ is equivalent to showing that

$$(i) \qquad\qquad (\tilde{E}_t \cap \{x > 0\}) \subset \bar{W} \quad \text{for all } t > \alpha(0).$$

The direction $\Rightarrow$ of the equivalence follows directly from the definitions (2.8), (2.3). The direction $\Leftarrow$ follows from the arguments of (2.6). Suppose, contrapositively, that there exist $x > 0$ and $\mathbf{p} \in \pi$, with $\mathbf{p} \cdot \mathbf{a} = x$ and $\alpha_1(\mathbf{p}) = \alpha(x) = t > \alpha(0)$. If any neighborhood of $\tilde{E}_t$ containing $\tilde{\mathbf{P}}_1(\mathbf{p})$ were contained in $\bar{W}$, the maximum principle would yield $\pi_t$-reflection symmetry, as in the proof of Lemma 2.6. But this is impossible since $\alpha(0) \neq t$.

To show (i) we introduce tilted planes. For small $\varepsilon > 0$ let $\pi^\varepsilon$ be the plane through the origin with normal $\mathbf{v}^\varepsilon$, a positive multiple of $\mathbf{v} - \varepsilon\mathbf{a}$. If $\pi$ is horizontal and $\mathbf{v}$ points upward, then $\pi^\varepsilon$ rises slightly as $x \to \infty$.

If we reflect $E \cap \Pi_t^\varepsilon$ through planes $\pi_t^\varepsilon$ the corresponding Alexandrov function $\alpha_1^\varepsilon$ must attain its maximum only on the *boundary* (2.5): since no plane parallel to $\pi^\varepsilon$ can be a plane of symmetry for $E$, $\alpha_1^\varepsilon$ has no interior maximum (2.6), and $\alpha_1^\varepsilon(x) \to -\infty$ as $x \to \infty$ because of the tilting.

Writing $z^\varepsilon$ for the (boundary) maximum value of $\alpha_1^\varepsilon$, it follows as above that the reflections $\tilde{E}_t^\varepsilon$ of $E \cap \Pi_t^\varepsilon$ through $\pi_t^\varepsilon$ satisfy

$$(ii) \qquad\qquad (\tilde{E}_t^\varepsilon \cap \{x \geq 2R\varepsilon\}) \subset \bar{W} \quad \text{for all } t \geq z^\varepsilon.$$

(The technical requirement $x \geq 2R\varepsilon$ implies that the projections of points in $\tilde{E}_t^\varepsilon$ to $\pi^\varepsilon$ are in the domain of $\alpha_1^\varepsilon$.)

By letting $\varepsilon \to 0$ it is easy to check that (ii) implies

$$(iii) \qquad\qquad (\tilde{E}_t \cap \{x > 0\}) \subset \bar{W} \quad \text{for all } t \geq \limsup_{\varepsilon \to 0} z^\varepsilon.$$

Because the $z^\varepsilon$ are boundary maximum values, semicontinuity (2.7) implies

(iv)
$$\limsup_{\varepsilon \to 0} z^\varepsilon \leq \alpha(0).$$

Combining (iii) and (iv) yields (i).

**(2.10) Theorem.** *If $\Sigma$ is a properly embedded, complete MCH-surface contained in a solid cylinder, then $\Sigma$ is rotationally symmetric with respect to a line parallel to the axis of the cylinder.*

*Proof.* If $\Sigma$ can be contained in a solid half cylinder $C_{a,R}^+(\mathbf{P})$, then by (1.8) $\Sigma$ is compact and hence a sphere. If $\Sigma$ extends to infinity in both directions of the cylinder, fix any plane $\pi$ containing the axis $\mathbf{a}$ of the cylinder. Apply (2.9) to both ends and conclude that either there is a $\pi$-parallel plane of symmetry or else the function $\alpha_1(\mathbf{p})$ has a local maximum at $\{x = 0\}$. In that case the $\pi$-parallel symmetry plane exists by (2.6). Therefore $\Sigma$ has symmetry planes parallel to every plane containing $\mathbf{a}$. But the center of mass of any cross-section of $\Sigma$ perpendicular to $\mathbf{a}$ must be contained in each symmetry plane. Hence all symmetry planes intersect in a line parallel to $\mathbf{a}$, and $\Sigma$ has rotational symmetry about this line.   q.e.d.

We can now confirm some conjectures of Meeks [9].

**(2.11) Theorem.** *If $\Sigma$ is a complete, properly embedded MCH-surface in $\mathbf{R}^3$ having two ends, it is a Delaunay surface. If $\Sigma$ has a finite number of ends and is contained in a half-space, say $\mathbf{R}^3 \backslash \Pi_0$ (where $\Pi_0$, $\pi$, and $\mathbf{v}$ are as in (2.1)), then it has a plane of symmetry parallel to $\pi$. Furthermore, $\Sigma$ is the union of a graph (of height at most $2H^{-1}$ above the symmetry plane) and its reflection.*

*Proof.* By Meeks' results any surface $\Sigma$ of the type above is the union of a compact subset with a finite number of cylindrically bounded ends, and if $\Sigma$ has only two ends it is contained in a cylinder (1.10), (1.12). In the latter case (2.10) implies the result.

In any case, if $\Sigma \subset \mathbf{R}^{n+1} \backslash \Pi_0$ has a finite number of cylindrically bounded ends we may consider the Alexandrov function $\alpha_1$ for $(\pi, \mathbf{v})$ on all of $\Sigma$. On any cylindrically bounded end which is *not* perpendicular to $\mathbf{v}$, $\alpha_1$ approaches $-\infty$. (That is, if for $\mathbf{p} \in \pi$ the corresponding $\mathbf{P}_1$ is far along one of those ends, its height above $\pi$ is very negative.) By applying (2.9) first and (2.6) if necessary, the $\pi$-parallel symmetry plane exists, as in the proof of (2.10).

The reflection surfaces $\tilde{\Sigma}_t$ are clearly graphs above $\pi_t$ until the first point of tangential contact with $\Sigma$ is attained. It follows that $\Sigma$ is the union of a

graph and its reflection. Our height estimate follows from a technical modification of the same estimate for compact graphs with planar boundary values (1.7). We leave the details to the reader.

**(2.12) Remark.** When proving the convergence of embedded ends to Delaunay surfaces in §5 we apply (2.11) to (two-dimensional) cylindrically-bounded surfaces that are only *weakly embedded*: $\Sigma$ will be a smooth immersion which is the boundary of a connected open domain $\Omega$. Thus $\Sigma$ fails to be embedded only by having tangential self-intersections between sheets with opposite mean curvature vectors.

For subsets $(S, W)$ of the weakly embedded pair $(\Sigma, \Omega)$ the Alexandrov function $\alpha_1$ can still be defined. The only difference is that if $P_1$ happens to be a point of surface self-intersection (with both sheets in $S$), we define $P_2 = P_1$. This preserves the relation between the Alexandrov function and reflection. The lemma (2.6) remains true, with no essential changes in the proof. In particular, our main results (2.9), (2.10), and (2.11) hold in this generality.

The following remark is not needed for the later sections in this paper. We include it for completeness and because it can be used to help prove the embeddedness of many immersed MC$H$-surfaces constructed by Kapouleas [8].

**(2.13) Remark.** Alexandrov reflection holds for MC$H$-immersions which extend appropriately to be immersions of their "interiors":

(i) *Configuration.* Let $S \subset \Sigma$, where $\Sigma$ is a constant mean curvature immersion. Suppose there exists an open bounded subset $W^{-1} \subset \mathbf{R}^{n+1}$, an immersion $\mathbf{F}: \bar{W}^{-1} \to \mathbf{R}^{n+1}$ and a smooth embedded surface (with boundary) $S^{-1} \subset \partial W^{-1}$ for which $\mathbf{F}(S^{-1}) = S$. Define $W = \mathbf{F}(W^{-1})$. Assume $S$ is *consistently oriented* with respect to $S^{-1}$. By this we mean that (for $H \neq 0$), the pull-back of $\mathbf{h}$ either always points into $W^{-1}$, or else always points out of it.

For $(\pi, \mathbf{v})$, $\mathbf{p} \in \pi$, we modify the definition of $\alpha_1$ (2.3). For our configuration (i), the $t$-values of $L_\mathbf{p} \cap \bar{W}$ are automatically bounded above and below. Consider $\mathbf{F}^{-1}(L_\mathbf{p} \cap \bar{W})$, the union of a countable collection of simple curves $\Gamma_i \subset W^{-1}$ with a set $\mathbf{Q} \subset \partial W^{-1}$. For each point $\mathbf{Q}_\beta \in \mathbf{Q}$ define $t_1 = t_2$ to be the $t$-value of $\mathbf{F}(\mathbf{Q}_\beta)$. For each curve $\Gamma_i$ define $t_1$ to be the supremum and $t_2$ to be the infimum of the $t$-values of $\mathbf{F}(\Gamma_i)$, and denote the corresponding boundary points of $\Gamma_i$ by $\mathbf{Q}_{i,1}$ and $\mathbf{Q}_{i,2}$. If, for the maximum value of $t_1 + t_2$ (taken over these subsets of $\mathbf{F}^{-1}(L_\mathbf{p})$), the corresponding $\mathbf{Q}_{i,1}$, $\mathbf{Q}_{i,2}$ (or $\mathbf{Q}_\beta$) are in $S^{-1}$, we say that $\mathbf{p}$ is in the domain

of the Alexandrov function $\alpha_1$ defined by

$$\alpha_1(\mathbf{p}) = \tfrac{1}{2} \max_{i,\beta}(t_1 + t_2).$$

Using this generalized $\alpha_1$ it is still easy to prove the Alexandrov reflection principle (2.6) and upper semicontinuity (2.7). The proofs are essentially unchanged. For the reflection principle, note that if $\mathbf{p}$ is an interior maximum of $\alpha_1$ (2.5), corresponding to points $\mathbf{Q}_{i,1}$, $\mathbf{Q}_{i,2}$ (or $\mathbf{Q}_\beta$) in $\bar{W}^{-1}$, then there will still be a sheet of $\tilde{S}_{\alpha_1(\mathbf{p})}$ making one-sided tangential contact with a sheet of $S$. The two sheets are images under $\mathbf{F}$ of neighborhoods of $\partial W^{-1}$ near $\mathbf{Q}_{i,1}$, $\mathbf{Q}_{i,2}$. The mean curvature vectors of the two sheets correspond at $\tilde{P}_2 = \mathbf{F}(\mathbf{Q}_{i,2})$ because of the consistent orientation assumption in (i). Semicontinuity follows as before except that one works with sequences of points in $\bar{W}^{-1}$ rather than in $\bar{W}$.

## 3. Weights and balancing, linear area growth, and monotonicity

Here we derive three basic facts about MC$H$-surfaces $\Sigma$ in Euclidean space (0.1). First, we obtain a *balancing formula*. It allows us to assign a *weight* vector to each end of an MC$H$-surface, and implies that the sum (over all ends) of these weights must vanish. We compute the weights in the axially symmetric case, and also in the case of certain minimal surfaces. Second, we prove that any cylindrically bounded end of $\Sigma$ has *linear area growth*, with growth rate depending on its weight. Third, we derive a *local area estimate* for $\Sigma$ using an extension of the well-known monotonicity formula.

Let $\Sigma \subset \mathbf{R}^{n+1}$ be a complete, properly embedded, $n$-dimensional surface. (We will restrict to MC$H$-surfaces later.) We consider (0.3) bounded open subsets $S \subset \Sigma$, $U \subset \mathbf{R}^{n+1}$, with (piecewise) smooth boundaries $\partial S$, $\partial U = S \cup Q$ $(\partial S = \partial Q)$. Let $\mathbf{h} = -h\nu$ be the mean curvature vector of $\Sigma$ (0.1). Let $\eta$ be the (exterior) conormal to $\partial S$, relative to $S$, and $\nu$ be the exterior normal to $\partial U$.

When $\mathbf{Y}$ is a smooth vector field on $\mathbf{R}^{n+1}$, write DIV $\mathbf{Y}$ for its divergence (0.1) on $\mathbf{R}^{n+1}$, and recall the relation to volume change as $U$ is deformed along $\mathbf{Y}$:

$$(3.1) \qquad \delta_{\mathbf{Y}}|U| = \int_U \mathrm{DIV}\,\mathbf{Y} = \int_{\partial U} \nu \cdot \mathbf{Y} = \int_S \nu \cdot \mathbf{Y} + \int_Q \nu \cdot \mathbf{Y}.$$

The analogous formula (see, e.g., [14]) for the rate of change of area as $S$ is deformed along $\mathbf{Y}$ is given by

$$(3.2) \qquad \delta_{\mathbf{Y}}|S| = \int_S \operatorname{div} \mathbf{Y} = \int_{\partial S} \eta \cdot \mathbf{Y} + \int_S h\nu \cdot \mathbf{Y},$$

where we have decomposed the divergence on $\Sigma$ into tangential and normal components $(0.1)$:

$$\operatorname{div} \mathbf{Y} = \operatorname{div}(\mathbf{Y}^\top + \mathbf{Y}^\perp) = \operatorname{div} \mathbf{Y}^\top + h\nu \cdot \mathbf{Y}.$$

(We have also used Stokes' theorem to represent the integral of the divergence $\operatorname{DIV} \mathbf{Y}$ in $(3.1)$—and of $\operatorname{div} \mathbf{Y}^\top$ in $(3.2)$—as a boundary integral.)

Combining $(3.1)$, $(3.2)$ we obtain (for any constant $H$) the *first variation formula*

$$(3.3) \qquad \begin{aligned} \delta_{\mathbf{Y}}(|S| - H|U|) &= \int_S \operatorname{div} \mathbf{Y} - H \int_U \operatorname{DIV} \mathbf{Y} \\ &= \int_{\partial S} \eta \cdot \mathbf{Y} - H \int_Q \nu \cdot \mathbf{Y} + \int_S (h - H)\nu \cdot \mathbf{Y}. \end{aligned}$$

Thus $\Sigma$ is an MC$H$-surface ($h \equiv H$) provided, for any bounded pair $(S, U) \subset (\Sigma, \mathbf{R}^{n+1})$ and any variation vector field $\mathbf{Y}$ which vanishes on $Q = \partial U \backslash S$,

$$\delta_{\mathbf{Y}}(|S| - H|U|) = 0.$$

Our formulas and estimates arise by computing the first variation of $|S| - H|U|$ for "geometrically interesting" vector fields $\mathbf{Y}$.

**(3.4) Theorem** (Weights and Balancing). *Let $\Sigma$ be an MC$H$-surface. Then (with the notation as above) we have the vector "balancing formula"*

$$(3.5) \qquad \int_{\partial S} \eta - H \int_Q \nu = \mathbf{0}.$$

*Moreover, there is a natural vector-valued cohomology class $\omega \in H^{n-1}(\Sigma) \otimes \mathbf{R}^{n+1}$ defined via the balancing formula $(3.5)$. Specifically, given a smooth $(n-1)$-cycle $\Gamma \subset \Sigma$, define the weight class $\omega$ on $[\Gamma] \in H_{n-1}(\Sigma)$ by*

$$(3.6) \qquad \omega([\Gamma]) = \int_\Gamma \eta - H \int_K \nu,$$

*where $K \subset \mathbf{R}^{n+1}$ is any smooth n-chain with $\partial K = \Gamma$. (Here the choices of normal $\nu$ to $K$ and conormal $\eta$ to $\Gamma$ are necessarily consistent with $(3.5)$.)*

*Proof.* If $\mathbf{Y}$ generates an isometry of $\mathbf{R}^{n+1}$, then $\delta_{\mathbf{Y}}|S|$ and $\delta_{\mathbf{Y}}|U|$ both vanish. Inserting the $(n+1)$ translation-generating vector fields $\mathbf{Y} = \mathbf{e}_i$ into $(3.3)$, expanding $0 = \delta_{\mathbf{Y}}(|S| - H|U|)$, and noting that $(h - H) \equiv 0$ on $S$ (since we *assume* $\Sigma$ is an MC$H$-surface) we obtain $(3.5)$.

The existence of the weight class $\omega \in H^{n-1}(\Sigma) \otimes \mathbf{R}^{n+1}$ follows from (3.5) and the usual methods for proving the isomorphism between *singular* and *deRham* cohomology. In fact, it is possible (using (3.6) and Stokes' theorem over $K$) to write explicitly a closed, vector-valued $(n-1)$-form on $\Sigma$ which upon integration over $\Gamma$ yields $\omega([\Gamma])$. We omit the computations.   q.e.d.

We use the preceding theorem to define the *weight vector* of a cylindrically-bounded end $E \subset \Sigma$ as follows.

**(3.7) Definition.** Let $\Sigma = \partial\Omega$ be an MC$H$-surface (0.1). Let $W = \Omega \cap \mathbf{C}_{\mathbf{a},R}^+(\mathbf{P})$, where $E = \partial W \backslash \mathbf{D}_{\mathbf{a},R}(\mathbf{P})$ is a cylindrically-bounded end (0.3). Let $\pi = \nu^\perp$ be an oriented hyperplane with $\pi \cap E$ transverse and compact, and with $\nu \cdot \mathbf{a} > 0$. Consider the homology class $[\pi \cap E] \in H_{n-1}(\Sigma)$. (This class is independent of $\pi$, because given acceptable planes $\pi_1$, $\pi_2$, one can find a third acceptable plane $\pi_3$ near infinity, so that, for $i = 1, 2$, $(\pi_3 \cap E) \cup (\pi_i \cap E)$ bounds a compact subset of $\Sigma$.)

The *weight* of $E$ is defined to be the vector $\mathbf{w}(E) = \omega([\pi \cap E])$. Explicitly:

$$(3.8) \qquad \mathbf{w}(E) = \int_{\pi \cap E} \eta - H \int_{\pi \cap W} \nu.$$

(To be consistent with (3.6) we choose the conormal $\eta$ on $\pi \cap E$ to satisfy $\nu \cdot \eta > 0$; both $\eta$ and $\nu$ "point toward infinity on $E$".)

**(3.9) Remark.** It follows from the balancing formula (3.5) that if the MC$H$-surface $\Sigma$ has a finite number of (cylindrically-bounded) ends, then the sum of their corresponding weights must be zero.

The weight (3.6), (3.8) has a nice interpretation for an axially symmetric MC$H$-surface $D$, i.e., a Delaunay surface. (We also include the case where $H = 0$, the catenoid.)

Consider the profile curve of $D$ in $\mathbf{R}^2$ (($x, r$)-space), given as the graph $r = \rho(x)$. Using a plane $\pi$ with normal $\nu = \mathbf{a}$ to compute the weight $\mathbf{w}$ of $E = \{\mathbf{p} \in D | x(\mathbf{p}) \geq 0\}$, we conclude from symmetry that $\mathbf{w}$ is parallel to $\mathbf{a}$. Hence

$$\mathbf{w} = m\mathbf{a} \quad \text{for a constant } m = \mathbf{a} \cdot \mathbf{w}, \text{ the mass of } E.$$

Observing that $\mathbf{a} \cdot \eta = (1 + (\rho')^2)^{-1/2}$ and $\mathbf{a} \cdot \nu = 1$, and also that $|\pi \cap E| = \rho^{n-1}|\mathbf{S}^{n-1}|$ and $|\pi \cap W| = \rho^n|\mathbf{B}^n|$, and recalling $|\mathbf{S}^{n-1}| = n|\mathbf{B}^n|$, we conclude from (3.8) that

$$(3.10) \qquad m \equiv \rho^{n-1}|\mathbf{B}^n| \left( n / \sqrt{1 + (\rho')^2} - H\rho \right).$$

Equation (3.10) can be thought of as a first order ODE for the profile of $D$, i.e. as a *first integral* of the second order ODE for $\rho$ which asserts that

$\Sigma$ has mean curvature $h = H$. (For the case $n = 2$, $H = 1$, the latter ODE is given by (5.5)(ii).) The solution to (3.10) can be expressed in terms of hyperelliptic functions [5], although the qualitative behavior is clear from (3.10) alone:

In the case $H \neq 0$, equation (3.10) implies the profile function $r$ is periodic. $D$ has alternating *necks* and *bulges* where $\rho'(x) = 0$, and we denote the corresponding radii by $\rho_-$ and $\rho_+$, respectively. From (3.10) it follows that

$$m = |\mathbf{B}^n|(\rho_\pm)^{n-1}(n - H\rho_\pm).$$

Fixing $H = 1$, we see that $m$ is maximized on the cylinder $\rho \equiv n - 1$, and that it tends monotonically to zero as $\Sigma$ approaches a chain of spheres ($\rho_- = 0$, $\rho_+ = n$).

Now consider the case $H = 0$. Then (3.10) implies $\rho'$ is increasing and so $D$ has a single neck, say, at $x = 0$; this lets one express the profile curve for $E$ as a graph above the $r$-axis, $x = x(r)$. Compute the mass from (3.8) by integrating the conormal alone, and conclude that

$$(3.11) \qquad x'(r)/\sqrt{1 + (x'(r))^2} = |\mathbf{S}^{n-1}|^{-1}mr^{1-n}.$$

Since $x'(r) \to 0$ as $r \to \infty$, $E$ is the graph over the plane $\mathbf{a}^\perp$ of an asymptotically harmonic function. In fact, upon integrating (3.11) for $r$ large, one concludes that $x$ has the asymptotic expansion

$$(3.12) \qquad x(r) \approx b + |\mathbf{S}^{n-1}|^{-1} \cdot \begin{cases} m \log r & (n = 2), \\ m(2 - n)^{-1}r^{2-n} & (n > 2), \end{cases}$$

for some constant $b$. Thus, in the case of minimal surfaces, one can view the mass as the "source strength" of the "potential" $x$.

If $M$ is a complete (immersed) minimal surface in $\mathbf{R}^3$ with *finite total curvature* (see §4 for definition) then it is a classical fact (see, e.g., [10]) that each embedded end of $M$ is asymptotic to a surface of revolution. (Compare our results in §§5 and 6 for the case $H \neq 0$.)

If this minimal $M$ is *embedded*, it follows that all its ends have the same axis vector $\mathbf{a}$. In the complement of $\mathbf{C}_{\mathbf{a},R}$ (for sufficiently large $R$) each end may be expressed as a graph over the plane $\mathbf{a}^\perp$ with the asymptotics (3.12) for $x$ and its derivatives holding (up to sign). (The same is true for $n > 2$, under the asymptotically-graphical assumption [12].) Furthermore we can order the ends according to their *height* $x = \mathbf{x} \cdot \mathbf{a}$. We will need the following result in §4.

**(3.13) Lemma.** *Let $M \subset \mathbf{R}^{n+1}$ be a connected, $n$-dimensional complete embedded minimal surface with each end satisfying the asymptotics (3.12).*

*Then* (*unless M is a plane*) *the top and bottom ends of M have nonzero weights, i.e., they are "catenoid" ends.*

*Proof.* It suffices to consider the top end $M^+$ and its weight vector $\mathbf{w}^+$. Let $b$ be the *least* value for which $M \subset \{\mathbf{x} \in \mathbf{R}^{n+1} | x \le b\}$. (We may assume $b < \infty$, for otherwise (3.12) implies $m > 0$.) The interior maximum principle implies either $x \equiv b$ or $x < b$ on $M$; in the latter case $M$ is asymptotic to the hyperplane $\{x = b\}$. For small enough $\varepsilon > 0$, let $M^\varepsilon$ be the (unique) noncompact component of $\{\mathbf{x} \in M^+ | x \ge b - \varepsilon\}$, and let $\eta$ be the conormal to $\partial M^\varepsilon$ which points into $M^\varepsilon$. By construction, $\eta \cdot \mathbf{a} \ge 0$. Since $M^\varepsilon$ is not flat, the boundary maximum principle implies that actually $\eta \cdot \mathbf{a} > 0$. Integrating $\eta \cdot \mathbf{a}$ over $\partial M^\varepsilon$, (3.8) implies $m = \mathbf{a} \cdot \mathbf{w}^+ > 0$.          q.e.d.

The second and third facts which we prove in this section are local area estimates for MC$H$-surfaces $\Sigma$. We derive them with the aid of the following "double" divergence formula, which is the right-hand side of (3.3) in case $h \equiv H$.

$$(3.14) \qquad \int_S \operatorname{div} \mathbf{Y} - H \int_U \operatorname{DIV} \mathbf{Y} = \int_{\partial S} \eta \cdot \mathbf{Y} - H \int_Q \nu \cdot \mathbf{Y}.$$

We now show that the area of a cylindrically bounded end $E$ on an MC$H$-surface $\Sigma$ grows linearly with length. For fixed axis $\mathbf{a}$ and radius $R$, we denote by $C_L$ the finite cylinder

$$C_L = \{x\mathbf{a} + \mathbf{D}_{\mathbf{a},R} | R \le x \le L - R\}.$$

**(3.15) Theorem** (Linear area growth). *Let $E \subset \mathbf{C}^+_{\mathbf{a},R}$ be a cylindrically bounded end on an MC1-surface $\Sigma \subset \mathbf{R}^{n+1}$. That is, $W = \Omega \cap \mathbf{C}^+_{\mathbf{a},R}$ and $E = \partial W \backslash \mathbf{D}_{\mathbf{a},R}$, as in (0.3). Let $\mathbf{w}$ be the weight of $E$. Then there exists a constant $c = c(n, R, \mathbf{a} \cdot \mathbf{w}) < \infty$ such that*

$$|E \cap C_L| < cL.$$

*Proof.* Fix $L \ge 2R$, and rotate so that $\mathbf{a} = (n+1)^{-1/2}(\mathbf{e}_1 + \mathbf{e}_2 + \cdots + \mathbf{e}_{n+1})$. For each $i = 1, \cdots, n+1$, we consider the "diagonally cut" segments

$$C^i = \{0 \le x^i \le La^i\} \cap \mathbf{C}_{\mathbf{a},R}.$$

For use in the double divergence formula (3.14), we define

$$U^i = C^i \cap W, \quad S^i = C^i \cap E, \quad Q^i = \partial C^i \cap W.$$

By construction we have

(i) $$(E \cap C_L) \subset \bigcap_{i=1}^{n+1} S^i.$$

Define $\mathbf{X}_i = x^i \mathbf{e}_i$ (no summation). Recall (or compute (0.1)) that

(ii) $\qquad \operatorname{div} \mathbf{X}_i = |\nabla x^i|^2 = 1 - (\nu^i)^2, \qquad \operatorname{DIV} \mathbf{X}_i = 1.$

Observe that $\pi \equiv \{\mathbf{x} \in \mathbf{R}^{n+1} | x^i = La^i\}$ is an acceptable plane for computing the weight of $E$ (3.8). (By Sard's Theorem, we may assume $\pi \cap E$ transverse, using a slightly shifted segment, if necessary.) Note also that $\mathbf{X}_i \equiv La^i \mathbf{e}_i$ on $\partial S^i \cap \pi$, and $\mathbf{X}_i \equiv \mathbf{0}$ on $\partial S^i \backslash \pi$. Inserting the vector field $\mathbf{X}_i$ into (3.14), and applying (ii) and (3.8), this immediately implies

(iii) $\qquad \displaystyle\int_{S^i} 1 - (\nu^i)^2 - |U^i| = La^i \mathbf{e}_i \cdot \mathbf{w}.$

Now sum equation (iii) from $i = 1$ to $n + 1$, apply (i), and conclude the estimate

$$n|E \cap C_L| \le \sum_{i=1}^{n+1} |U^i| + L\mathbf{a} \cdot \mathbf{w}.$$

Because $U^i \subset C^i$, its volume $|U^i|$ is bounded by a multiple of $L$. Hence this estimate implies the theorem.  q.e.d.

Finally we discuss the (almost) area monotonicity of MCH-surfaces. This is a simple extension of the minimal surface monotonicity formula, but does not seem to explicitly appear in the literature.

**(3.16) Proposition** (*Monotonicity formula*).  *Let $\Sigma = \partial \Omega \subset \mathbf{R}^{n+1}$ be a complete, properly embedded MCH-surface. Let $S(r) = \Sigma \cap \mathbf{B}_r(\mathbf{P})$. Then*

$$\frac{d}{dr}(r^{-n}|S(r)|) \ge -H|\mathbf{S}^n|,$$

*and for $0 < r < 1$,*

$$r^{-n}|S(r)| \le |S(1)| + H|\mathbf{S}^n|(1 - r).$$

*Proof.* We assume $\mathbf{P} = \mathbf{0}$ and insert the position field $\mathbf{X} = \sum_{i=1}^{n+1} x^i \mathbf{e}_i$ into our double divergence formula (3.14), integrating over $S(r)$, $U(r) = \Omega \cap \mathbf{B}_r$ and $Q(r) = \Omega \cap \mathbf{S}_r$. Since $\operatorname{div} \mathbf{X} = n$ and $\operatorname{DIV} \mathbf{X} = n + 1$ (see (3.15)(ii)), and since $\mathbf{X} \cdot \nu \equiv r$ on $Q(r)$, (3.14) yields

(i) $\qquad n|S(r)| - (n + 1)H|U(r)| = \displaystyle\int_{\partial S(r)} \mathbf{X} \cdot \eta - rH|Q(r)|.$

It is geometrically clear (and follows from the co-area formula) that

(ii) $\qquad \displaystyle\int_{\partial S(r)} \mathbf{X} \cdot \eta \le r\frac{d}{dr}|S(r)|.$

Making the substitution of (ii) into (i), discarding the $rH|Q(r)|$ term and estimating $|U(r)|$ by $|\mathbf{B}_r|$, we obtain

(iii) $$r\frac{d}{dr}|S(r)| - n|S(r)| \leq -(n+1)H|\mathbf{B}_r|.$$

Multiply (iii) by $r^{-n-1}$. Note that $(n+1)|\mathbf{B}_1| = |\mathbf{S}_1|$, and deduce the differential inequality claimed in (3.16). Integrate (3.16) between $r$ and $1$ to derive the subsequent estimate.

(3.17) **Remark.** The main results of this section can be generalized in several directions. Since they only really depend on the first variation formula (3.3), they generalize to $MCH$-surfaces with self-intersections and singularities. Specifically, one can regard $\Sigma$ as an *integral n-current* represented as the boundary of the $(n+1)$-current $\Omega$; $\Sigma$ has (generalized) constant mean curvature $H$ if and only if

$$\delta_\mathbf{Y}(|\Sigma| - H|\Omega|) = 0$$

for all compactly supported variations $\mathbf{Y}$.

One can show, for instance, that any *immersed* $MCH$-surface has well-defined weights, these must "balance" in the sense of (3.4), and—for ends asymptotic to the self-intersecting Delaunay surfaces—the corresponding masses $m = \mathbf{a} \cdot \mathbf{w}$ turn out to be *negative*.

These ideas generalize to other ambient manifolds as well. Take, for example, the cohomological interpretation of weights (3.6): if one identifies the *Killing fields* on the ambient manifold $N$ with the *Lie algebra* $\mathbf{g}$ of its *isometry group* $\mathbf{G}$, then to any $n$-dimensional $MCH$-surface $\Sigma \subset N$ one can assign the *moment class*

$$\mu \in H^{n-1}(\Sigma) \otimes \mathbf{g}^*$$

with coefficients in the dual Lie algebra $\mathbf{g}^*$. (Even in $\mathbf{R}^{n+1}$ one can glean more information about the structure of $MCH$-surfaces by using the *rotation-generating vector fields*, which correspond to the $\mathbf{so}(n+1)$ factor in the Lie algebra of the Euclidean group.)

## 4.   Uniform curvature estimate

In this section we show that a properly embedded $MCH$-surface $\Sigma \subset \mathbf{R}^3$ has uniformly bounded curvature. We begin by using the Gauss-Bonnet formula and linear area growth (3.15) to show that $\int |A|^2$ also grows linearly on each annular end $A \subset \Sigma$. The rest of the proof is indirect: if $|A|$ were unbounded on $A$, then a sequence of suitably scaled copies of $A$ converges to a (nonflat) complete embedded minimal surface with finite

total curvature; but in this event the weight formula (3.6) for $A$ is violated, a contradiction.

Without loss of generality we will assume $\Sigma = \partial\Omega$ is an MC1-surface, and focus on a fixed annular end $A \subset \Sigma$. Applying Meeks' Theorem (1.10) we find (after a possible translation) a cylindrically-bounded end $E \subset A$, $E = \partial W \backslash \mathbf{D}_{\mathbf{a},R}$, where $W = \mathbf{C}^+_{\mathbf{a},R} \cap \Omega$. Let $x(\mathbf{P}) = \mathbf{P} \cdot \mathbf{a}$ denote the coordinate function along the axis $\mathbf{a}$. For any $U \subset \mathbf{R}^+$ define $E_U = \{\mathbf{P} \in E | x(\mathbf{P}) \in U\}$.

We call a connected open set $\mathscr{O} \subset E_{(a,b)}$ *critical* if $x(\mathscr{O}) = (a,b)$. We need the following preparatory lemma which is a consequence of the height-4 estimate (1.7) and an innermost-loop argument. (We also use this lemma in §5.)

**(4.1) Lemma.**  *If $(b-a) > 4$ then there exists a unique critical component $\mathscr{O} \subset E_{(a,b)}$. In fact, if $a > 4$, there exists an annulus $Z$ such that $E_{(a+4,b-4)} \subset \mathscr{O} \subset Z \subset E_{(a-4,b+4)}$.*

*Proof.*  We may (by Sard's Theorem, slightly shifting the original interval if necessary) assume that $\mathbf{F}^{-1}(E_{\{a\}} \cup E_{\{b\}}) \subset \mathbf{D}\backslash\mathbf{0}$ (0.3) is a finite union of disjoint simple loops. Call each an *a-loop* or *b-loop* according to its image under $\mathbf{F}$. Such a loop either winds once around $\mathbf{0}$ and is *essential*, or else it is *trivial* on $\mathbf{D}\backslash\mathbf{0}$.

The height-4 estimate (1.7) implies that the F-image of any compact region on $\mathbf{D}\backslash\mathbf{0}$ which is bounded entirely by $a$-loops lies in $E_{(a-4,a+4)}$. The corresponding statement holds for $b$-loops. In particular, no component of $E_{(a,b)}$ whose inverse image lies inside a trivial loop can be critical.

Now study the essential loops, which are nested. Call the one closest to $\mathbf{0}$ *innermost*. Since $x = 0$ on $\partial E$ and $x \to \infty$ as we approach $\mathbf{0}$ on the disk, the innermost curve is a $b$-loop and the outermost is an $a$-loop. From innermost to outermost, all must be $b$-loops until the first $a$-loop, after which all are $a$-loops. Otherwise one would have a compact region bounded by two $b$-loops whose F-image extends to $E_{(0,a)} \subset E_{(0,b-4)}$ violating (1.7).

Let $Z$ be the F-image of the annular domain between the innermost $a$-loop and the outermost $b$-loop. Assessing the picture described above, it follows that the only critical component of $E_{(a,b)}$ is the planar domain $\mathscr{O} \subset Z \cap E_{(a,b)}$ whose boundary contains $\partial Z$. In fact, each remaining component of $E_{(a,b)} \backslash \mathscr{O}$ is bounded entirely by $a$-loops or $b$-loops and cannot extend into $E_{(a+4,b-4)}$ by (1.7). Therefore $E_{(a+4,b-4)} \subset \mathscr{O}$. The same argument implies $Z \subset E_{(a-4,b+4)}$.

**(4.2) Theorem.**  *There is a constant $d = d(A)$ so that for $L \geq d$ we have*

$$\int_{E_{(L,2L)}} |\mathbf{A}|^2 \leq dL.$$

*Proof.* First observe that $|\mathbf{A}|^2 = H^2 - 2K$ (writing $K$ for the Gauss curvature), so

$$(4.3) \qquad \int_S |\mathbf{A}|^2 = |S| - 2 \int_S K$$

on any compact portion $S$ of an MC1-surface. Thus to translate linear area growth (3.15) into linear $\int |\mathbf{A}|^2$ growth, it suffices to find a portion $S \subset E$ comparable to $E_{(L,2L)}$ on which $|\int_S K|$ is uniformly bounded.

Let $c$ be a bound (3.15) for the maximum amount of area in any segment $E_{(x,x+1)} \subset E$. Then we can find a segment $E_{(a,b)}$ as in (4.1) so that

(i)  $\qquad\qquad\qquad |E_{\{a\}}|, |E_{\{b\}}| < 2c,$

(ii)  $\quad (a + 5 + c, b - 5 - c) \subset (L, 2L) \subset (a + 4 + c, b - 4 - c).$

Let $Z \subset E_{(a-4,b+4)}$ be the annulus constructed in (4.1). Choose geodesic loops $G_a$, $G_b$ which minimize length among essential loops in $A$ based at fixed points $\mathbf{P}_a \in \partial Z \cap E_{\{a\}}$, $\mathbf{P}_b \in \partial Z \cap E_{\{b\}}$, respectively. Note that (i) implies $|G_a|, |G_b| < 2c$, and it follows from (ii) and (4.1) that

(iii)  $\qquad\qquad\qquad E_{(L,2L)} \subset S \subset E_{(a-4-c,b+4+c)},$

where $S$ is the annulus bounded by $G_a$ and $G_b$. We use this $S$ in (4.3).

Since the total geodesic curvature of $\partial S$ is majorized by $2\pi$, with a (singular) contribution of at most $\pi$ from each of the corners $\mathbf{P}_a$ and $\mathbf{P}_b$, the Gauss-Bonnet formula implies $|\int_S K| \le 2\pi$ as well. Thus (4.3), (ii), (iii) and (3.15) imply

$$\int_{E_{(L,2L)}} |\mathbf{A}|^2 \le \int_S |\mathbf{A}|^2 \le |E_{(a-4-c,b+4+c)}| - 2\int_S K \le c(L + 18 + 4c) + 4\pi.$$

For suitable choice of $d$, this yields (4.2).   q.e.d.

Next we state two general results for embedded hypersurfaces of bounded curvature.

**(4.4) Uniform graph lemma.** *Let $M^n \subset \mathbf{R}^{n+1}$ with normal $\nu$. Let $R > 0$ and $\mathbf{p} \in \mathbf{R}^{n+1}$. Suppose that $M \cap \mathbf{B}_{2R}(\mathbf{p})$ is closed (no boundary), embedded and has uniformly bounded curvature, $|\mathbf{A}| < C$. Then there exists $r = r(C, R)$ such that for all $\mathbf{P} \in M \cap \mathbf{B}_r(\mathbf{p})$ the $\mathbf{P}$-containing component of $M \cap \mathbf{C}_{\nu(\mathbf{P}),r}(\mathbf{P})$ is a graph above $\mathbf{D}_{\nu(\mathbf{P}),r}(\mathbf{P})$ of a function $u$ (0.2), such that $|\mathbf{D}^2 u| \le 2C$, $|\mathbf{D}u| \le 2Cr$, and $|u| \le Cr^2$.*

*Proof.* This lemma follows from the fact that the second fundamental form $\mathbf{A}$ is expressible in terms of $\mathbf{D}u$ and $\mathbf{D}^2 u$, with $\mathbf{A} = \mathbf{D}^2 u$ when $\mathbf{D}u = 0$. We leave the details to the reader.   q.e.d.

**(4.5) Parallel-sheets lemma.** *Let $M^n \subset \mathbf{R}^{n+1}$ with normal $\nu$. Let $R > 0$ and $\mathbf{p} \in \mathbf{R}^{n+1}$. Suppose that $M \cap \mathbf{B}_{2R}(\mathbf{p})$ is closed, embedded and has uniformly bounded curvature, $|\mathbf{A}| < C$. Then there exists $K = K(R, C)$ so that for all $\mathbf{P}_1, \mathbf{P}_2 \in M \cap \mathbf{B}_R(\mathbf{p})$, we obtain (after possibly changing the sign of $\nu(\mathbf{P}_2)$):*

$$|\nu(\mathbf{P}_1) - \nu(\mathbf{P}_2)| \leq K|\mathbf{P}_1 - \mathbf{P}_2|^{1/2}.$$

*Proof.* The existence of some modulus of continuity for $|\nu(\mathbf{P}_1) - \nu(\mathbf{P}_2)|$ is clear geometrically, and that is all we really need in the sequel. The details are somewhat tedious, however, so we omit them. To understand why the correct modulus is actually $|\mathbf{P}_1 - \mathbf{P}_2|^{1/2}$, consider the following example with $n = 1$. Let $M^1 \subset \mathbf{R}^2$ as above with $M^1 \cap \mathbf{B}$ equal to a line segment almost tangent to a disjoint circular arc. If the segment is exactly tangent to the arc, then near the tangency it is easy to check that the normals of the arc converge to the normal of the segment at a rate proportional to the square root of the distance between corresponding points on the arc and segment. q.e.d.

We now state and prove the main result of this section.

**(4.6) Theorem.** *A properly embedded MCH-surface $\Sigma \subset \mathbf{R}^3$ has uniformly bounded curvature. That is, on any annular end $A \subset \Sigma$, there exists $C < \infty$ so that $|\mathbf{A}| < C$ on $A$.*

*Proof.* Our strategy is to derive a contradiction by assuming that $|\mathbf{A}|$ is unbounded on a fixed cylindrically-bounded end $E \subset A$. We use a "blow up" argument.

**(4.7)** *If $|\mathbf{A}|$ is unbounded on $E$ we can find a sequence of points $\{\mathbf{P}_k\} \subset E$ together with a sequence of radii $\{\rho_k\} \subset (0, 1]$ such that*

$$\mu_k \equiv 2\rho_k \cdot |\mathbf{A}(\mathbf{P}_k)| \to \infty \quad \text{as } k \to \infty,$$

$$|\mathbf{A}(\mathbf{q})| \leq 2|\mathbf{A}(\mathbf{P}_k)| \quad \text{for all } \mathbf{q} \in \mathbf{B}_{\rho_k}(\mathbf{P}_k) \cap E.$$

*Proof of (4.7).* Define the piecewise linear function $\eta_k$ by:

(i) $$\eta_k(t) = \begin{cases} 1, & 0 \leq t \leq k, \\ (k+1) - t, & k < t \leq k+1, \\ 0, & k+1 < t. \end{cases}$$

Find $\mathbf{P}_k \in E$ with $x_k = \mathbf{P}_k \cdot \mathbf{a}$ so that

(ii) $$\eta_k(x_k)|\mathbf{A}(\mathbf{P}_k)| = \max_{\mathbf{P} \in E} \eta_k(x)|\mathbf{A}(\mathbf{P})| \equiv \mu_k.$$

Our hypothesis implies $a_k \equiv |\mathbf{A}(\mathbf{P}_k)| \to \infty$ and $\mu_k \to \infty$ as $k \to \infty$. Because $|\nabla \eta_k| \leq 1$,

(iii) $$|\mathbf{A}(\mathbf{Q})| < 2|\mathbf{A}(\mathbf{P}_k)| \quad \text{for all } \mathbf{Q} \in E \text{ s.t. } |\mathbf{Q} - \mathbf{P}_k| < \eta_k(x_k)/2.$$

Defining $\rho_k = \eta_k(x_k)/2$, we obtain (4.7).

For each of the points $\mathbf{P}_k$ above, consider the scaling transformation of $\mathbf{R}^3$ defined by $\mathbf{x} \to a_k(\mathbf{x} - \mathbf{P}_k)$. Let $E_k$ denote the image of $E$ under this scaling, and let $\mathbf{A}_k$ be its second fundamental form. In view of claim (4.7) above we have

$$(4.8) \qquad |\mathbf{A}_k(\mathbf{0})| = 1, \qquad |\mathbf{A}_k(\mathbf{q})| \leq 2 \quad \text{for all } \mathbf{q} \in E_k \cap \mathbf{B}\mu_k.$$

The area and monotonicity estimates (3.15), (3.16), and the total curvature bound (4.2), combined with the behavior, respectively, of area and curvature under scaling, imply

$$(4.9) \qquad |E_k \cap \mathbf{B}(\mathbf{q})| \leq C \quad \text{for all } |\mathbf{q}| < \frac{a_k}{2} - 1,$$

$$(4.10) \qquad \int_{E_k \cap \mathbf{B}_{a_k}} |\mathbf{A}|^2 \leq C,$$

where $0 < C < \infty$ is a constant depending on the original surface $E$, but not on $k$. The estimates (4.8), (4.9), (4.10), together with the Uniform graph lemma (4.4) and the Parallel-sheets lemma (4.5), allow us to extract a well-behaved convergent subsequence, as follows.

Associate to each $E_k$ its corresponding *multiplicity*-1 *rectifiable varifold*; that is, in the notation of [14, Chapter 4], we consider the sequence of varifolds $\{V_k\}$, where $V_k \equiv \mathbf{v}(E_k, 1)$. We use varifolds mainly to handle multiplicity in the limit; although we could use W. Allard's varifold compactness theorem [14, Chapter 8] to obtain convergence, it is simple to argue directly here.

**(4.11)** *A subsequence of* $\{V_k\}$ *converges to a limit varifold* $V = \mathbf{v}(M, \mathfrak{m})$ *where* $M$ *is a nonflat, complete embedded minimal surface in* $\mathbf{R}^3$ *with (finite) integer multiplicity* $\mathfrak{m}$ *and finite total curvature*:

$$0 < \int_M |\mathbf{A}_M|^2 < \infty.$$

*Proof of* (4.11). Choose $r$ corresponding to $R = 1$ and $C = 2$ in the Uniform graph lemma (4.4). Partition $\mathbf{R}^3$ with parallel cubes $\{Q_i\}$ of diameter $r/4$. For given $k$ and $Q_i \subset \mathbf{B}_{\mu_{k-1}}$ (see (4.7)) choose points $\mathbf{x}_{i,k,l} \in E_k$, with normals $\nu_{i,k,l}$ so that (using multi-index notation: $(i, k, l) \equiv \alpha$) the corresponding graphs of $u_\alpha$ above $\mathbf{D}_{\nu_\alpha, r/2}(\mathbf{x}_\alpha)$ cover $E_k \cap Q_i$. By the area estimate (4.9) the index $l$ can be assumed uniformly bounded in each $Q_i$. By duplication if necessary, we assume the index $l$ runs from 1 to $l(i)$ independently of $k$.

Using Cantor diagonalization and Heine-Borel we extract a subsequence such that the $\mathbf{x}_\alpha$ and $\nu_\alpha$ converge as $k \to \infty$ (uniformly, for $i$ bounded). By

elliptic compactness we may also assume that the $u_\alpha$ converge (uniformly and smoothly above the discs of radius $r/2$) to solutions of the minimal surface equation (and uniformly so, for $i$ bounded).

This strong "sheetwise" local convergence immediately implies that the corresponding varifold subsequence $\{V_k\}$ converges to a limiting varifold $V$, for which the modulus of continuity ascribed to the normals of each $E_k$ by (4.5) persists. Invoking the strong maximum principle to rule out any self-tangency, we conclude that $V = \mathbf{V}(M, \mathfrak{m})$, where $M \subset \mathbf{R}^3$ is a complete embedded minimal surface, and $0 < \mathfrak{m} \in \mathbf{Z}$ is a locally constant multiplicity function; $\mathfrak{m}$ is finite because of the local area bound (4.9). Similarly, the curvature estimate (4.10) persists in the limit, so that $M$ has finite total curvature. Finally, $M$ is nonflat since (4.8) and the smooth convergence imply $|\mathbf{A}_M(\mathbf{0})| = 1$. This establishes (4.11).

We complete the proof of the theorem by comparing the weight vectors of $M$ and $E$. Let $\mathbf{w}$ be the weight of the annular end $E \subset A$ (3.8). Align $M$ (and $E$) so that all the ends of $M$ have limiting normals parallel to $\mathbf{e}_3$ at infinity. Study the top end $M^+$ of $M$. By (3.13) the weight vector $\mathbf{w}^+ = m\mathbf{e}_3$ of $M^+$ is nonzero.

Let $\gamma$ be the curve of (transverse) intersection of $M$ with the $\{x^3 \equiv t\}$ plane, where $t$ is chosen so large that the conormal $\eta$ is almost constant along the almost circular $\gamma$. (This is possible by (3.12). The approximate radius $R$ of $\gamma$ and approximate "inclination angle" $\eta^3$ are related by $2\pi R\eta^3 \approx m$.)

By the arguments of (4.11), each $E_k$ separates locally into annular sheets which converge smoothly to $M$ near $\gamma$ as $k \to \infty$. For each $k$ we choose a component $\gamma_k$ of $E_k \cap \{x^3 \equiv t\}$, with $\gamma_k \to \gamma$ smoothly.

Let $\varepsilon_k = 1/a_k$ (in the notation of (4.7)) and observe that, being the inverse image of $\gamma_k$ under the previously used scaling transformation, the curve $\Gamma_k = \mathbf{P}_k + \varepsilon_k\gamma_k$ lies on $E$. Using the planar cap $K_k$ spanning $\Gamma_k$, we compute the weight vector $\mathbf{w}_k = \omega([\Gamma_k])$ assigned to the homology class $[\Gamma_k]$ on $E \subset A$ (3.6):

$$(4.12) \qquad \mathbf{w}_k = \int_{\Gamma_k} \eta - \int_{K_k} \nu = \varepsilon_k(\mathbf{w}^+ + \mathbf{o}(\varepsilon_k)) - \mathbf{O}(\varepsilon_k^2) = \varepsilon_k\mathbf{w}^+ + \mathbf{o}(\varepsilon_k).$$

The error term of order $\mathbf{O}(\varepsilon_k^2)$ in (4.12) arises from the area $|K_k|$ of the cap $K_k$; the correction of order $\varepsilon_k\mathbf{o}(\varepsilon_k)$ occurs in the conormal integral because the sheets of $E_k$ are converging to $M$ smoothly.

Since $\mathbf{w}^+ \neq \mathbf{0}$, the right-hand side of (4.12) cannot stay within $\mathbf{o}(\varepsilon_k)$ of any finite set of vectors as $\varepsilon_k \to 0$. But this is a contradiction: by (3.6) $\mathbf{w}_k$

can attain at most three values: $\pm\mathbf{w}$, when $[\Gamma_k]$ is essential on $A$ (the sign depending on the orientation), or $\mathbf{0}$, when $[\Gamma_k]$ is trivial on $A$.

## 5.  Strong asymptotic convergence of embedded ends

In this section we show that any embedded annular end $A \subset \mathbf{R}^3$ of an MC$H$-surface $\Sigma$ converges exponentially to a fixed Delaunay surface in $\mathbf{R}^3$. It is relatively easy, using the results of the previous sections, to show that compact subsets of $A$ near infinity are approximately Delaunay. Furthermore, the Delaunay surfaces which approximate $A$ in this way are all translations of a particular solution $D$ along its axis. To prove that $A$ actually converges exponentially to a fixed translation of $D$, we study Jacobi fields on $D$ arising from the approximation process.

We normalize the mean curvature so that $\Sigma$ is an MC1-surface. Then, as in §4, we apply Meeks' Theorem (1.10) to isolate a cylindrically bounded end $E = A \cap \mathbf{C}_{\mathbf{a},R}^+$.

**(5.1)** For any sequence $\{t_k\}$ with $t_k \to \infty$, and for the cylindrically bounded end $E \subset \mathbf{C}_{\mathbf{a},R}^+$ define the *slide-back sequence* $\{E^k\}$ for $E$ by $E^k = E - t_k\mathbf{a}$.

**(5.2) Theorem.**  *Let $E \subset \mathbf{C}_{\mathbf{a},R}^+$ be as above. Then any slide-back sequence for $E$ has a convergent subsequence in $\mathbf{R}^3$. The limit is a Delaunay surface $D$ having the same weight as $E$. The convergence may be taken to be $C^2$ on compact subsets of $D$, viewing the $E^k$ as normal graphs above $D$. All convergent slide-back sequences converge to translations of $D$ along its axis line $L_D$.*

*Proof.*  We use essentially the same compactness argument developed in (4.11): since $E$ has uniformly bounded curvature (4.6), we can partition $\mathbf{R}^3$ with cubes $\{Q_i\}$ and construct uniformly-sized graphical coordinate patches. The linear area growth estimate (3.15) then controls the number of "coordinate patches" in any particular $Q_i$. Hence in the limit (of a subsequence which we also index with $k$) we again get an integer multiplicity rectifiable varifold $V$, with support equal to the union of uniformly smooth graphs above discs, with each $\mathbf{P} \in V$ in the interior of at least one of these graphs, and with a well-defined (up to sign) limiting "normal" $\nu_\infty(\mathbf{P})$. (Note that spt $V$ is nonempty: for any $c$, the "cross-section" $E^k \cap \{x \equiv c\}$ is nonempty and uniformly bounded for large $k$, so (spt $V$) $\cap \{x \equiv c\}$ is also nonempty.)

It remains to study the convergence of $\{E^k\}$ near arbitrary $\mathbf{P} \in$ spt $V$. The same sheeting property used in proving (4.11) holds in this context;

for all $\varepsilon > 0$ there exists $\delta = \delta(E) > 0$ so that for $k$ large enough we have:

(i)     Any component $E^{k,i}$ of $E^k \cap (\mathbf{P} + \mathbf{C}\nu_{\infty(\mathbf{P}),2\delta})$ which intersects $\mathbf{B}_\delta(\mathbf{P})$ is the graph of a function $u^{k,i}$ above $\mathbf{P} + \mathbf{D}\nu_{\infty(\mathbf{P}),2\delta}$, with $|\mathbf{D}u^{k,i}| < \varepsilon$.

Because $E^k$ bounds a region, the normal $\nu$ changes orientation between successive sheets $E^{k,i}$. As before, extract a subsequence with $1 \leq i \leq N$ independently of $k$ and with $u^{k,i+1} \geq u^{k,i}$ for $1 \leq i < N$. Each $u^{k,i}$ satisfies the nonparametric mean curvature equation, with mean curvature alternately $+1$, $-1$ as $i$ varies:

(ii)     $$\mathrm{DIV}\left(\frac{\mathbf{D}u^{k,i}}{\sqrt{1 + |\mathbf{D}u^{k,i}|^2}}\right) = \pm 1.$$

Assume $u^{k,i}$ and $u^{k,i-1}$ satisfy (ii) with right-hand sides $-1$ and $1$ respectively. Expanding (ii) for each, taking their difference and using our bounds for their first and second derivatives, we conclude that on a disc of radius $2\delta$, the nonnegative function $u^{k,i} - u^{k,i-1}$ satisfies $\Delta(u^{k,i} - u^{k,i-1}) = -2 + O(\varepsilon)$. Choosing $\varepsilon > 0$ sufficiently small, we may assume that $\Delta(u^{k,i} - u^{k,i-1}) < -1$. Comparing with the function $v = (4\delta^2 - |\mathbf{x}|^2)/2$ and applying the maximum principle, we conclude that

(iii)     $u^{k,i} - u^{k,i-1} > \delta^2 \quad$ above $\mathbf{P} + \mathbf{D}_{\nu_{\infty(\mathbf{P}),\delta}}$.

As in §4 we now pass to subsequences that converge in $C^2$ above $\mathbf{P} + \mathbf{D}_{\nu_{\infty(\mathbf{P}),\delta}}$ to solutions $u^{\infty,i}$ of the same equation. Then $u^{\infty,i_0}$ contains $\mathbf{P}$ for some $1 \leq i_0 \leq N$, and we may assume that $u^{\infty,i_0}$ satisfies (ii) with $-1$ on the right-hand side. By (iii) the only other function $u^{\infty,i}$ whose graph may contain $\mathbf{P}$ is then $u^{\infty,i_0+1}$, in which case their mean curvature vectors are opposite. Hence in a uniform neighborhood of $\mathbf{P}$ we have either one or two distinct coordinate patches for one or two smooth immersions. Hence spt $V$ represents one or more smooth, complete, cylindrically bounded, weakly embedded (in the sense of (2.13)) submanifolds having constant mean curvature (with the mean curvature vector pointing *into* the open set bounded by each submanifold (0.1)). By (2.13) and the results of §2, each of these submanifolds is then a Delaunay surface or sphere.

From the arguments above one now deduces that on any compact subset of $\mathbf{R}^3$, and once $k$ is large enough, spt $V$ is given as a (normal) graph above $E^k$ with small $C^2$ norm, and vice versa. But then spt $V$ contains no spheres, because $A$ has only one component. By the critical component lemma (4.1) it also follows that spt $V$ may not contain more than one Delaunay surface. Hence spt $V$ is a Delaunay surface $D$, in fact covered only once, and the

convergence is $C^2$ on compact subsets of $D$ (i.e. the function expressing $E^k$ as a normal graph above this compact subset of $D$ converges to zero in $C^2$ as $k \to \infty$).

Given this local $C^2$ convergence, the weight vector (3.8) of $D$ must be that of $A$. Along with the cylindrical boundedness of $A$, this determines the limiting Delaunay surface of any convergent slide-back sequence up to translation. But the locally smooth convergence also implies (using the axis line $L_D$ of the limit $D$) that Alexandrov functions (2.9) for planes containing $L_D$ all decay monotonically to zero. Hence the axis of every limiting Delaunay surface is $L_D$.

**(5.3) Definition.** For an axis vector $\mathbf{a}$, an orthonormal frame $\{\mathbf{b}, \mathbf{c}\}$ for $\mathbf{a}^\perp$, and a function $\rho(x, \theta)$, the image of the mapping

$$\mathbf{F}(x, \theta) = x\mathbf{a} + \rho(x, \theta)\omega(\theta),$$

with $\omega(\theta) = \mathbf{b}\cos\theta + \mathbf{c}\sin\theta$, is called the *cylindrical graph* of the function $\rho$. Generally we will arrange $(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$, in which case $\mathbf{F}(x, \theta) = (x, \rho\cos\theta, \rho\sin\theta)$. We measure the *steepness* of the graph via the natural functional $v$:

$$v = \frac{1}{\nu \cdot \omega} = \sqrt{1 + \rho_x^2 + (\rho_\theta/\rho)^2},$$

$\nu$ being, as usual, the "outer" normal.

**(5.4) Justified assumptions.** For the remainder of this section we fix a particular Delaunay surface $D$ with axis vector $\mathbf{a}$, one which arises from some slide-back sequence (5.2).

We assume that $E \subset \mathbf{C}_{\mathbf{a},R}^+$ is the cylindrical graph of a function $\rho_E$ above the axis of $D$. Let $D$ have neck radius $r_-$ and bulge radius $r_+$ as in (3.10). We assume that $\rho_E$ is bounded between $\frac{1}{2}r_-$ and 2, that the steepness (5.3) of $E$ is bounded by twice the maximum steepness of $D$, and that the curvature $|\mathbf{A}_E|$ is less than twice the maximum of $|\mathbf{A}_D|$. These assumptions are justified by Theorem (5.2): if no finite slide-back of the original end satisfied these bounds we could obtain a sequence so that, after sliding back, the bounds were always violated on the same compact subset of $\mathbf{R}^3$. Such a sequence could not converge in a locally $C^2$ manner to any translation of $D$, violating (5.2). Hence some finite slide-back of the original end is satisfactory as our new end $E$.

We further assume that all derivatives of $\rho$ are bounded uniformly for $x \geq 0$. This assumption is justified by our uniform curvature estimate, the higher order elliptic estimates which it implies [3], and by our assumptions on the bounds for $\rho$ and the steepness $v$.

We now recall the elliptic partial differential equation satisfied by a function $\rho$ parametrizing an MC1 surface, and the ordinary differential

equation satisfied by radially symmetric (Delaunay) solutions. The former is uniformly elliptic by (5.4); $v$ is given by (5.2):

(5.5)(i)
$$\left(\frac{1 + (\rho_\theta/\rho)^2}{v^3}\right)\rho_{xx} + \left(\frac{1 + \rho_x^2}{\rho^2 v^3}\right)\rho_{\theta\theta} - \left(\frac{2\rho_x\rho_\theta}{\rho^2 v^3}\right)\rho_{x\theta}$$
$$- \frac{1}{\rho v^3}(1 + \rho_x^2 + 2(\rho_\theta/\rho)^2) + 1 = 0,$$

(5.5)(ii)
$$\frac{\rho_{xx}}{v^3} - \frac{1}{\rho v} + 1 = 0.$$

(The quickest way for the reader to rederive (5.5) is to use the parametrization (5.2). Recall that in general if one has a hypersurface parametrization in Euclidean space given by $\mathbf{F}(y)$ one computes its mean curvature by first computing the metric tensor $[g_{ij}] = [\partial\mathbf{F}/\partial y^i \cdot \partial\mathbf{F}/\partial y^j]$; then the mean curvature is given by the trace, $g^{ij}A_{ij}$, where $[g^{ij}]$ is the inverse matrix to $[g_{ij}]$.)

If $\rho$ solves (5.5)(i), then by a straightforward computation, $w \equiv \rho - \rho_D$ satisfies

(5.6) $\left(\dfrac{\rho^2}{v^2}\right)w_{xx} + w_{\theta\theta} + \left(\dfrac{\rho\rho_x}{v^2} - 3\dfrac{\rho^2\rho_x\rho_{xx}}{v^4}\right)w_x + w = Q(w, \mathrm{D}w, \mathrm{D}^2w).$

Here $Q$ is a sum of terms, each uniformly smooth (depending on D) and depending at least quadratically on $(w, \mathrm{D}w, \mathrm{D}^2w)$. Furthermore, terms involving $\mathrm{D}^2w$ always contain a factor of $w$ or $\mathrm{D}w$.

Let $\tilde{I} \subset\subset I \subset (-\infty, \infty)$. Let $|w|_{C^0(I)} = \varepsilon$. (We suppress the $\theta$ dependence when we write expressions for norms.) By the Fundamental Theorem of Calculus and the uniform bound on $|\mathrm{D}^2w|$ (5.4) it follows that $|\mathrm{D}w| \leq C\varepsilon^{1/2}$ on $\tilde{I}$. Writing $Q(x, \theta)$ for $Q(w, \mathrm{D}w, \mathrm{D}^2w)|_{(x,\theta)}$, and applying (5.4) to estimate second and third derivatives of $w$, we obtain two estimates:

(5.7) $\qquad |Q(x,\theta)|_{C^0(\tilde{I})} \leq C \cdot (|w|_{C^0(I)})^{1/2}, \qquad |Q(x,\theta)|_{C^{1/2}(\tilde{I})} \leq C.$

From standard linear elliptic regularity theory [3] it now follows from (5.6) and (5.7) that we have the estimate $|w|_{C^2(\tilde{I})} \leq C \cdot (|w|_{C^0(I)})^{1/2}$. Recycling this estimate through (5.6) and applying elliptic estimates again we see that (if we actually start with an intermediate subinterval)

(5.8) $\qquad\qquad\qquad |w|_{C^3(\tilde{I})} \leq C|w|_{C^0(I)}.$

It will be natural in our arguments to use the following norm:

(5.9) $\qquad\qquad ||w||_I \equiv \sup_{x\in I}\left(\int_0^{2\pi} w(x,\theta)^2\, d\theta\right)^{1/2}.$

Because we always deal with $w$'s having uniformly bounded second derivatives it is easily seen that $||w||_{\{x\}}$ is uniformly equivalent to $|w|_{C^0(\{x\})}$ (with the equivalence depending on the second derivative bounds), hence from (5.8) we conclude

$$(5.10) \qquad |w|_{C^3(\tilde{I})} \leq C||w||_I.$$

**(5.11) Lemma.** *Let $\tilde{I} \subset\subset I \subset\subset (-\infty, \infty)$. Let $\{E^k\}$ be a slide-back sequence. Let $\rho_k(x, \theta) = \rho_E(x + t_k, \theta)$. Suppose that for the functions $w_k = \rho_k - \rho_D$, $||w_k||_I \equiv \varepsilon_k \to 0$ as $k \to \infty$. Define $u_k \equiv w_k/\varepsilon_k$. Then a subsequence of $\{u_k\}$ converges in $C^2(\tilde{I})$ to a Jacobi field $u$, i.e., a solution of the Jacobi equation:*

$$(5.12) \qquad Lu = \left(\frac{\rho^2}{v^2}\right) u_{xx} + u_{\theta\theta} + \left(\frac{\rho\rho_x}{v^2} - \frac{3\rho^2\rho_x\rho_{xx}}{v^4}\right) u_x + u = 0.$$

*Proof.* By (5.10) the sequence $\{u_k\}$ is bounded in $C^3(\tilde{I})$. Hence it is compact in $C^2(\tilde{I})$, and a convergent subsequence may be chosen. Because of the quadratic nature of $Q$ in (5.6), and by the estimate (5.10) it follows that $Q(w_k, Dw_k, D^2w_k)/\varepsilon_k \to 0$. Hence (5.6) reduces to (5.12).    q.e.d.

The following lemma follows from the monotonicity of the Alexandrov function (2.9). It will control the behavior of $\rho_k - \rho_D$ and corresponding Jacobi fields near infinity.

**(5.13) Lemma.** *There exists $C < \infty$ so that for sufficiently small $\varepsilon > 0$ the assumption that $\sup|\rho_\theta| < \varepsilon$ at $x = b$ implies that $|\rho_\theta| < C\varepsilon$ for all $x \geq b$.*

*Proof.* We show that the bound on $|\rho_\theta|$ at $x = b$ yields the same order estimate on $\alpha(b)$ for all Alexandrov functions $\alpha(x)$ relative to planes $\pi$ containing **a**. By monotonicity of $\alpha$ (2.9) the estimate then holds for all $x \geq b$. This will imply that $|\rho_\theta| < C\varepsilon$ for such $x$.

Pick an Alexandrov function $\alpha$ as above. Orient $\mathbf{R}^3$ so that $\pi$ is horizontal, so that $\mathbf{v} = \mathbf{e}_3$ points upward and so that $(\mathbf{a}, \mathbf{b}, \mathbf{c}) = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ in the parametrization (5.3). Recall the points $\mathbf{P}_1 = \mathbf{p} + t_1\mathbf{v} \equiv \mathbf{F}(x, \theta_1)$ and $\mathbf{P}_2 = \mathbf{p} + t_2\mathbf{v} \equiv \mathbf{F}(x, \theta_2)$ used to define $\alpha_1$ (2.3) when $\mathbf{p} \in \pi$ and $L_\mathbf{p} \cap E$ is nonempty. For such $\mathbf{p}$ it follows that

$$(i) \qquad \alpha(x) = \frac{1}{2} \max_{\mathbf{p}\cdot\mathbf{e}_1 = x} \{\rho(x, \theta_1)\sin\theta_1 + \rho(x, \theta_2)\sin\theta_2\}.$$

Let $\Gamma$ denote the (almost-circular) intersection curve of $E$ with the plane $x \equiv b$, parametrized by $\mathbf{F}(b, \theta) = (b, \rho\cos\theta, \rho\sin\theta)$ (5.3). Then $\alpha(b)$ is attained, corresponding to points $\mathbf{P}_1, \mathbf{P}_2 \in \Gamma$ as above, and the reflection of

$\Gamma$ through the plane of height $\alpha(b)$ is tangent to itself at $\mathbf{P}_2$. In particular, the unit tangent vectors to $\Gamma$ have *opposite* $\mathbf{e}_2$ components at $\mathbf{P}_1$ and $\mathbf{P}_2$:

(ii)
$$\left.\frac{\rho \sin \theta - \rho_\theta \cos \theta}{\sqrt{\rho^2 + \rho_\theta^2}}\right|_{\theta_1} = \left.\frac{\rho_\theta \cos \theta - \rho \sin \theta}{\sqrt{\rho^2 + \rho_\theta^2}}\right|_{\theta_2}.$$

Rearranging (ii) and using the bounds (5.4) on $\rho$ and $\rho_\theta$, we estimate

$$\rho(b, \theta_1) \sin \theta_1 + \rho(b, \theta_2) \sin \theta_2 \leq C\varepsilon,$$

so that $\alpha(b) \leq C\varepsilon$ (i). This estimate holds for all oriented planes $\pi$ containing $\mathbf{a}$, and by (2.9), for all $x \geq b$.

Conversely, we show that if $\alpha(x) \leq \varepsilon$ for all oriented planes $\pi$ containing $\mathbf{a}$, then there exists $C < \infty$ so that $|\rho_\theta(x)| \leq C\varepsilon$. For $\mathbf{P} \in E$ with $P^1 = x$, pick $(\pi, \mathbf{v})$ and coordinates as above so that $P^3 > 0$ and so that the tangent plane of $E$ at $\mathbf{P}$ is vertical (parallel to $\mathbf{v} = \mathbf{e}_3$). Then $\alpha_1(x, P^2, 0) \geq P^3$. Also, $(\rho \cos \theta)_\theta = 0$ at $\mathbf{P}$:

(iii)
$$\rho \sin \theta = \rho_\theta \cos \theta.$$

By the assumptions on the steepness of $E$ (5.4), $\theta$ is bounded away from $\pi/2$ at this "vertical" point, so there exists $\mu > 0$ with $|\cos \theta| > \mu$. Since $P^3 = \rho \sin \theta$, our estimate for $\alpha$ and (iii) imply that $|\rho_\theta| < C\varepsilon$ with $C = \mu^{-1}$.

Combining the estimates of the two preceding paragraphs we conclude that (5.13) holds for small $\varepsilon > 0$.

**(5.14) Lemma.** *Let $[a, b] = \tilde{I} \subset\subset I \subset\subset (-\infty, \infty)$. Then the corresponding sequence $\{u_k\}$ defined in Lemma (5.11) has a subsequence which converges (locally) in $C^2$ norm to a Jacobi field defined on the entire interval $[a, \infty)$.*

*Proof.* By estimate (5.10), Lemma (5.11) and Cantor Diagonalization it suffices to first fix $a' < a$ with $[a', b] \subset\subset I$, and then to show that for any $M < \infty$ there exists $C = C(M)$ so that $|w_k|_{C^0([a', M])} \leq C\varepsilon_k$, where $w_k$ and $\varepsilon_k$ are as in (5.11).

We write $O(\varepsilon_k)$ for quantities that are bounded in norm by a $k$-independent multiple of $\varepsilon_k$. By (5.10) $|Dw_k|$ is $O(\varepsilon_k)$ at $a'$. In particular $|(w_k)_\theta| = |(\rho_k)_\theta|$ is $O(\varepsilon_k)$ there. By Lemma (5.13) this estimate holds for all $x \geq a'$. But $(\rho_k)_\theta$ satisfies a uniformly elliptic equation on the entire interval $[a', \infty)$. (Differentiate equation (5.5)(i) with respect to $\theta$.) Hence by standard (interior) linear elliptic theory (for $x > b$) $|(\rho_k)_\theta|_{C^1[a', \infty)}$ is $O(\varepsilon_k)$. Therefore we can consider the equation (5.5)(i) as a small ($O(\varepsilon_k)$) perturbation of (5.5)(ii), noting that (uniformly for fixed $\theta$) the initial conditions

at $x = a'$ for $\rho_k$ and $\rho_D$ also differ by $O(\varepsilon)$. Hence by the continuous-dependence-on-parameters theorem for ordinary differential equations, the solutions differ by $O(\varepsilon_k)$ on any finite interval $[a', M]$. (In fact the "$O(\varepsilon_k)$" depends exponentially on $(M - a')$.)   q.e.d.

We now separate variables to find an $L^2$ basis of solutions to (5.12). Writing $u(x, \theta) = X(x)\Theta(\theta)$ and assuming that $\Theta$ is $2\pi$-periodic, we find the separated equations

$$(5.15)(\text{i}) \qquad \Theta_{\theta\theta} + k^2\Theta = 0, \qquad k = 0, 1, 2, \cdots,$$

$$(5.15)(\text{ii}) \qquad X_{xx} - \left(\ln\left(\frac{v^3}{\rho}\right)\right)_x X_x + (1 - k^2)\frac{v^2}{\rho^2}X = 0.$$

We now need to analyze the solutions of (5.15)(ii), in particular their growth. For $k = 0$ we do this geometrically. Let $\{D_s\}_{0 < s \leq 1}$ be the one-parameter family of MC1-Delaunay surfaces having necks of radius $s$ at $x = 0$. (We can assume that $D = D_\sigma$ in this family.) Since each $D_s$ is the cylindrical graph of a $\theta$-independent function $\rho^s$ satisfying (5.5)(ii), we observe that $u(x, \theta) \equiv \frac{d}{ds}(\rho^s)|_{s=\sigma}$ is a $\theta$-independent Jacobi field, hence solves (5.15)(ii) with $k = 0$. Denote this solution by $X_{0,D}$. In the case where $D$ is a cylinder ($\sigma = 1$) we can also embed it in the family of Delaunay surfaces having bulges at $x = 0$, obtaining a second, linearly independent solution. (In fact, the two solutions are then $\sin(x)$ and $\cos(x)$.)

If $D$ is *not* the cylinder, we obtain a second solution by considering the one-parameter family $D_s = D + se_1$ of coaxial translations. Reasoning as above we then find that the partial derivative $(\rho_D)_x$ satisfies (5.15)(ii) with $k = 0$, and is independent of $X_{0,D}$.

When $k > 0$ we study (5.15)(ii) by making a change of independent variable $t = g(x)$, where $g'(x) = \rho^{-1}v^3$. This eliminates the first order term in (5.15)(ii), which transforms to

$$(5.16) \qquad X_{tt} + \frac{(1 - k^2)}{v^4}X = 0.$$

Note that the growth properties of $X(t)$ and $X(x)$ are equivalent since the ratio $t/x$ is bounded between positive constants.

We immediately see that when $k = 1$, $X$ must be of the form $C_1 + C_2 t = C_1 + C_2 g(x)$; i.e., there is a unique (up to scalar multiple) bounded solution ($X \equiv 1$), and all other solutions with $k = 1$ grow linearly. We remark that "geometrically", the solutions 1 and $g(x)$ describe infinitesimal translations of $D$ perpendicular to its axis, and rotations of $D$ about an axis normal to its own, respectively.

When $k \geq 2$, there always exists (up to scalar multiple) exactly one nonzero solution of (5.15)(ii) which decays exponentially to zero as $x \to \infty$. All other solutions are bounded below in absolute value by $e^{\lambda x}$ for some $\lambda > 0$ as $x \to \infty$. To show this it suffices to construct one solution in each of these categories. Construct the first via a shooting argument: The sequence of (necessarily positive) solutions to the two point boundary value problem with $X(0) = 1$, $X(M) = 0$ increase monotonically to the desired solution as $M \to \infty$. One can show the solution decreases exponentially by using barriers to the approximate solutions; namely, positive exponential functions with value 1 at the origin and with rate of decay slower than the minimum value of $(k^2 - 1)^{1/2} v^{-2}$. It is also straightforward to show that the solution with $X(0) = 1$, $X'(0) = 1$ grows exponentially.

**(5.17) Lemma.** *If the Jacobi field $u$ on $[a, \infty)$ arises from slide-backs, as in Lemma (5.14), then $u$ has the expansion*

$$u(x, \theta) = a_0 (\rho_D)_x + \sum_{k=2}^{\infty} \sin(k\theta + \phi_k) X_k(x),$$

*where, for $k \geq 2$, $\phi_k$ is constant and $X_k$ is a (possibly vanishing) exponentially decaying solution of (5.15)(ii), as described above. That is, up to the $(\rho_D)_x$ term, $u$ decays at a uniform exponential rate as $x \to \infty$.*

*Proof.* Separating variables in the standard way, we know the Jacobi field $u$ has an expansion

(i) $$u = \sum_{k=0}^{\infty} \sin(k\theta + \phi_k) X_k(x),$$

which converges to it, locally in $C^2$. The significance of our claim is that we have specified the $k = 0$ term, asserted the $k = 1$ term is zero, and ruled out all of the unbounded $X_k(x)$.

To show that no $X_k(x)$ can be unbounded we note the fact that $|u_\theta|$ is uniformly bounded on $[a, \infty)$, an immediate consequence of (5.13). Hence the $L^2$ integral of $u$ with respect to $\theta$, on $x =$constant cross sections, is also uniformly bounded. We calculate this integral by differentiating the series (i):

(ii) $$\int_{x=b} u_\theta^2 \, d\theta = \frac{1}{\pi} \sum_{k=1}^{\infty} k^2 X_k^2(b),$$

and conclude that all $X_k$'s are bounded for $k \geq 1$. By our discussion above it follows that $X_1$ is constant and that for $k \geq 2$, $X_k$ is exponentially decreasing.

To show that $X_1 = 0$ we use the $C^2$ approximation which arises from the Jacobi field construction; recalling the notation and results of (5.11) and (5.14) we may write, for any fixed interval,

$$\text{(iii)} \qquad \rho_j = \rho(x + t_j, \theta) = \rho_D + \varepsilon_j u + o(\varepsilon_j),$$

where $o(\varepsilon_j)/\varepsilon_j \to 0$ as $j \to \infty$. If $X_1 \neq 0$ we may choose $\mathbf{e}_2$ and $\mathbf{e}_3$ so that $\phi_1 = 0$ and $X_1 < 0$ in the expansion (i). Consider the Alexandrov functions $\alpha_1$ and $\alpha$ on $E$ for the plane spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$, with normal $\mathbf{v} = \mathbf{e}_3$. Note that $\rho_D + \varepsilon_j X_0$ is axially symmetric and that the $X_k$ terms decay for $k \geq 2$. Hence by considering a sufficiently long interval $[0, L]$ in (iii) we conclude that $\alpha_1((t_j + L)\mathbf{e}_1)$ is negative for large $j$. This contradicts the fact, as used in the proof of Theorem (5.2), that $\alpha$ decays monotonically to zero.

Finally, we show that $X_0$ is a multiple of $(\rho_D)_x$. By our discussion above we may write $X_0 = a_0(\rho_D)_x + b_0 X_{0,D}$. Approximate the weight vector of $E$ (3.8) using (iii). The $\varepsilon_j a_0 (\rho_D)_x$ term contributes nothing to the weight approximation, and by the periodicity of the sine function, neither do the $k \geq 1$ terms from $\varepsilon_j u$'s expansion. Because any nonzero multiple $X_{0,D}$ *does* contribute, it follows that $b_0$ must equal zero.

**(5.18) Main Theorem.** *For each end $E$ of the embedded MCH-surface $\Sigma$ there is a Delaunay surface $D$ in $\mathbf{R}^3$ to which it converges exponentially. That is, near infinity, $E$ and $D$ can be expressed as cylindrical graphs of functions $\rho_E$ and $\rho_D$ respectively, with $|\rho_E - \rho_D| < Ce^{-\lambda x}$ ($C, \lambda > 0$ constants) as $x \to \infty$.*

*Proof.* We are guided by the argument in [15]. Let $D$ be one of the (unique up to coaxial translation) Delaunay limits for $E$ near infinity, as in Theorem (5.2). Define $\rho^t(x, \theta) \equiv \rho_E(x + t, \theta)$ and $w^t \equiv \rho^t - \rho_D$. It will suffice to find a (large) positive multiple $T$ of the Delaunay surface's period and a bound $0 < N < \infty$ so that the following *approximation improvement property* holds:

(i)      Whenever $t$ is sufficiently large and $\varepsilon(t) \equiv \|w^t\|_{[0,T]}$ is sufficiently small, we have $\varepsilon(t + T + s) < \varepsilon(t)/2$ for some $s = s(t), |s| < N\varepsilon(t)$.

It is relatively straightforward to show that this approximation improvement property implies the desired exponential decay $w^\sigma \to 0$ for some translation $\sigma$ as follows. Using Theorem (5.2) and an initial slide-back, we may assume without loss of generality that (i) holds whenever

$t \geq 0$ and $\varepsilon(t)$ is sufficiently small, that $\varepsilon(0)$ *is* sufficiently small and that $N\varepsilon(0) < T/2$. Beginning with $t_0 = 0$, $s_0 = s(t_0)$, recursively define, for $j \in \mathbf{Z}^+$,

$$\text{(ii)} \qquad \sigma_j = \sum_{i=0}^{j-1} s_i, \quad t_j = t_{j-1} + s_{j-1} + T = \sigma_j + jT, \quad s_j = s(t_j),$$

and estimate from (i)

$$\text{(iii)} \qquad\qquad \varepsilon(t_j) < 2^{-j}\varepsilon(0), \qquad |s_j| < 2^{-j-1}T.$$

We claim that $\sigma = \lim_{j \to \infty} \sigma_j$ is a translation for which $w^\sigma \to 0$. For $x > 0$, write $x = x' + jT$, $x' \in [0, T)$, $j \in \mathbf{Z}$. Using the periodicity of $\rho_D$ with respect to $T$ and (ii), express

$$
\begin{aligned}
w^\sigma(x, \theta) &= \rho_E(x + \sigma, \theta) - \rho_D(x) \\
&= (\rho_E(x + \sigma_j, \theta) - \rho_D(x)) + (\rho_E(x + \sigma, \theta) - \rho_E(x + \sigma_j, \theta)) \\
&= w^{t_j}(x', \theta) + (\rho_E(x + \sigma, \theta) - \rho_E(x + \sigma_j, \theta)).
\end{aligned}
$$

Using the triangle inequality, the equivalence of the sup and $\|\cdot\|$ norms, the estimates (iii) and the uniform boundedness of $(\rho_E)_x$ (5.4), we estimate

$$|w^\sigma(x, \theta)| < C2^{-j}, \qquad |w(x, \theta)| < 2Ce^{-(\ln 2/T)x},$$

as desired.

It therefore remains to establish the approximation improvement property for some choice of multiperiod $T$ and bound $N$. We do this by observing what happens if (i) does *not* hold for particular choices of $T$ and $N$. In this case there exists a sequence $t_j \to \infty$ with $\varepsilon_j \equiv \varepsilon(t_j) = \|w^{t_j}\|_{[0,T]} \to 0$, but, $\varepsilon(t_j + T + s) > 2^{-1}\varepsilon_j$ whenever $|s| \leq N\varepsilon_j$. We set $u_j = \varepsilon_j^{-1} w^{t_j}$, and apply Lemmas (5.14), (5.17) to obtain a Jacobi field $u$ to which the $u_j$'s converge (locally in $C^2$) for $x \geq 1$:

$$\text{(iv)} \qquad u(x, \theta) = a_0(\rho_D)_x + \sum_{k=2}^{\infty} \sin(k\theta + \phi_k) X_k(x).$$

We note that by construction $\|u\|_{[T,2T]} \geq 1/2$, so that $u$ is not identically zero. If $D$ is the cylinder we may take $a_0 = 0$. Otherwise, the estimate $\|u\|_{[0,T]} \leq 1$ implies that there is an upper bound (depending inversely on the maximum value of $(\rho_D)_x$) on all possible values of $|a_0|$. If the choice of $N$ for (i) is greater than this upper bound, then the "correction translation" $s_j \equiv -\varepsilon_j a_0$ satisfies $|s_j| < N\varepsilon_j$.

Use the smooth, uniform convergence of $u_j$ to $u$ on the $x$-interval $[1, 2T]$ and (iv) to estimate:

$$w^{t_j + s_j}(x, \theta) = \rho_E(x + t_j - \varepsilon_j a_0, \theta) - \rho_D(x)$$
$$= w^{t_j}(x, \theta) - a_0 \varepsilon_j (\rho_E)_x + o(\varepsilon_j),$$

(v)     $$w^{t_j + s_j}(x, \theta) = \varepsilon_j \left( \sum_{k=2}^{\infty} \sin(k\theta + \phi_k) X_k(x) \right) + o(\varepsilon_j).$$

Since the series in (v) decays exponentially to zero as $x \to \infty$ (and always with a uniform rate), sufficiently large choice of $T > 1$ ensures

$$\|w^{t_j + s_j + T}\|_{[0,T]} = \|w^{t_j + s_j}\|_{[T, 2T]}$$

$$= \varepsilon_j \max_{x \in [T, 2T]} \left( \frac{1}{\pi} \sum_{k=2}^{\infty} X_k^2(x) \right)^{1/2} + o(\varepsilon_j)$$

(vi)

$$\leq \frac{\varepsilon_j}{3} \left( \frac{1}{\pi} \sum_{k=2}^{\infty} X_k^2(x) \right)^{1/2} + o(\varepsilon_j)$$

$$\leq \frac{1}{3} \|w^{t_j}\|_{[0,T]} + o(\varepsilon_j).$$

Assessing (vi) we see that if $N$ is chosen to bound all possible $|a_0|$ as above, and if $T$ is chosen sufficiently large, then the *negation* of (i) could not hold. Hence (i) holds for such $N$ and $T$, and Theorem (5.18) holds as well.

**(5.19) Corollary.** *Any complete, properly embedded $MC1$ $\Sigma \subset \mathbf{R}^3$ with finite type is conformally diffeomorphic to a compact Riemann surface having finitely many punctures.*

*Proof.* This is an easy exercise when $\Sigma$ is Delaunay: parametrize conformally with respect to polar coordinates $(r, \theta)$ in the plane so that the axial and radial coordinates $x$ and $\rho$ of $\Sigma$ are both functions of $r$ alone. The differential equation (3.10) then becomes

$$\frac{dx}{dr} = \frac{1}{2r} \left( \frac{m}{\pi} + \rho^2 \right).$$

The parenthesized quantity is bounded between nonzero constants (3.10ff), so $x$ tends logarithmically to $\pm\infty$ as $r \to 0, \infty$. Hence Delaunay surfaces are conformally punctured planes, i.e., twice-punctured spheres.

More generally, each end $E \subset \Sigma$ is *asymptotically* Delaunay as described in our Main Theorem (5.18). By standard uniformization theory, $E$ is conformally diffeomorphic to exactly one of the annuli

$$A(r_0) := \{ z \in \mathbf{C} \colon r_0 < |z| < 1 \},$$

with $0 \leq r_0 < 1$. We must show that, in fact, we always have $r_0 = 0$.

Let $\rho_E$ and $\rho_D$ be as in our Main Theorem, which gives the estimate $|\rho_E - \rho_D| = O(e^{-\lambda x})$ for some $\lambda > 0$. Since both functions satisfy (5.5)(i), which is uniformly elliptic under our justified assumptions (5.4), this estimate actually holds in $C^k$ for every $k$, by standard elliptic theory [3, 16.7]. So by deleting $E \cap B_R$ for some $R > 0$, we clearly arrange that the radial projection from $E$ onto the corresponding end of $D$ is $K$-quasiconformal for some $K \in [1, \infty)$ [17, 13.1, 15.1]. But having already seen that the ends of $D$ are conformally $A(0)$, this produces a $K$-quasiconformal map from $A(r_0)$ to $A(0)$, the ratio of whose conformal moduli would then be finite and nonzero if $r_0 \neq 0$. Since $A(0)$ has conformal modulus $\infty$ [17, 7.5], we are done.

**(5.20) Remark.** Starting from our main theorem above, and using the arguments of [16, claim 3.3] one can show more precisely that

$$\rho_E - \rho_D = X_k \sin(k\theta + \phi_k) + o(X_k),$$

where $X_k$ is a (necessarily nonvanishing) exponentially decaying solution to (5.15)(ii) for some $k \geq 2$. In particular, $E$ converges to $D$ at least as fast as a decreasing $X_2$ decays to zero.

## 6. Concluding remarks

**(6.1)** For ease of exposition we have restricted our attention to complete properly embedded surfaces $\Sigma$. Our Main Theorem (5.18) actually holds for any properly embedded MC1-annulus, not just subsets of complete, embedded surfaces. Only technical modifications to the ancillary theorems are needed for this more general result.

**(6.2)** It is natural to ask which results of this paper remain true in higher dimensions. In particular, under what hypotheses may one conclude that some embedded end of an MC1-hypersurface converges smoothly to one of the "Delaunay" hypersurfaces mentioned in §3. That section and its predecessor are written for hypersurfaces in $\mathbf{R}^{n+1}$. The height-4 estimate (1.7) and its corollaries (1.8), (4.1) are $n$-dimensional, as is much of §5: if the assumption (5.4) holds, then the subsequent Jacobi-field arguments which culminate in Main Theorem (5.18) have $n$-dimensional analogs.

The main obstruction to higher dimensional generalization is the lack of any sort of cylindrical boundedness estimate. By example (1.6), if such a result remains true for annular ends in $\mathbf{R}^{n+1}$, its proof is more complicated. Of course, in higher dimensions surfaces of finite type (0.3) need not have annular ends: an example in $\mathbf{R}^4$ is the product of a cylinder in $\mathbf{R}^3$ with a line.

Even if one assumes the cylindrical boundedness of an annular end there is a problem in trying to prove the analog of the slide-back theorem (5.2). One can "slide back" such an end, even take a (varifold or current) limit, but existing regularity theorems control neither the convergence, nor the limit itself. Without smoothness, our techniques for producing asymptotically rotation-invariant subsequences cannot get started.

(6.3) It would be interesting to clarify the asymptotic behavior of *immersed* $MCH$-surfaces. Recall (cf. §3) that for an immersed minimal surface of finite total curvature one knows that each embedded end is asymptotic to a surface of revolution. The following example, based on the H. Wente construction [18], shows that immersed ends need *not* be asymptotically rotationally symmetric.

Following Wente, one constructs a doubly periodic harmonic map $\mathbf{R}^2 \rightarrow \mathbf{R}^3$, which is realized as the Gauss map of the MC1-surface. In this realization, the period lattice in $\mathbf{R}^2$ is carried into a rank-2 discrete abelian subgroup of the group of Euclidean motions on $\mathbf{R}^3$. The subgroup is generated by a pair of screw-motions about a common axis $\mathbf{a}$. Choosing (as Wente did) a rectangular period lattice (and suitable boundary conditions on the fundamental rectangle), one generator is a rotation and the other is a translation. By reversing the closing-procedure used by Wente to construct the torus, one first arranges the rotation angle to be a rational multiple of $2\pi$, so the surface is an immersed cylinder, and then lets the translation remain nonzero to ensure that this immersed cylinder is proper.

We note that, generically, the Wente construction produces a complete immersed $\mathbf{R}^2 \subset \mathbf{C}_{\mathbf{a},R}$, but that the immersion is not proper. It would be interesting to construct (or rule out) a properly immersed MC1-plane in $\mathbf{R}^3$. It is also interesting to ask what natural assumptions, if any, on a properly immersed MC1-surface $\Sigma \subset \mathbf{R}^3$ imply that each end of $\Sigma$ is asymptotically a Wente-Delaunay cylinder.

(6.4) There seems to be a strong analogy between (intrinsic) scalar-Ricci curvature and (extrinsic) mean curvature-second fundamental form problems. The best known example of this is the correspondence between the results of R. Hamilton for the flow of a metric by the Ricci-curvature [4] and those of G. Huisken for the flow of a hypersurface by its mean curvature vector [6]. Similarly, the constant mean curvature constructions of N. Kapouleas [7], are related to analogous results of R. Schoen for conformally flat constant scalar curvature metrics [13]. An analog of our Main Theorem (5.18) for the conformally-flat scalar curvature problem would be that if a positive smooth solution $u$ to

(i) $$\Delta u + u^{(n+2)/(n-2)} = 0 \qquad (n > 2)$$

in the punctured ball $\mathbf{B}^n \backslash 0$ has a nonremovable singularity at $\mathbf{0}$, then $u$ converges strongly to one of the rotationally symmetric "Delaunay" solutions of (i). It seems likely that methods analogous to ours can be used to prove such a result.

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